Math 131BH - Weeks 8-9
Textbook pages covered: 299-314

- The problem of measure
- Additivity, sub-additivity, countable additivity
- Outer measure
- Outer measure is not additive
- Measurable sets; Lebesgue measure
- Measurable functions
- Simple functions

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Measure and integral

- Last week we discussed differentiation in several variable calculus. It is now only natural to consider the question of integration in several variable calculus. The general question we wish to answer is this: given some subset Ω of $\mathbf{R}^n$, and some real-valued function $f : \Omega \to \mathbf{R}$, is it possible to integrate f on Ω to obtain some number $\int_\Omega f$? (It is possible to consider other types of functions, such as complex-valued or vector-valued functions, but this turns out not to be too difficult once one knows how to integrate real-valued functions, since one can integrate a complex or vector valued function, by integrating each real-valued component of that function separately).

- In one dimension we already have developed (in Math 131A) the notion of a *Riemann integral* $\int_{[a,b]} f$, which answers this question when Ω is an interval $\Omega = [a, b]$, and f is *Riemann integrable*. Exactly what Riemann integrability means is not important here, but let us just remark that every piecewise continuous function is Riemann integrable, and in particular every piecewise constant function is Riemann integrable. However, not all functions are Riemann integrable. It is possible to extend this notion of a Riemann integral to higher dimensions, but it requires quite a bit of effort and one can still only integrate "Riemann integrable" functions, which turn out to be a rather unsatisfactorily small class of functions. (For instance, the pointwise limit of Riemann integrable functions need not be Riemann integrable, and the same goes for an L^2 limit, although we have already seen that uniform limits of Riemann integrable functions remain Riemann integrable).

- Because of this, we must look beyond the Riemann integral to obtain a truly satisfactory notion of integration, one that can handle even very discontinuous functions. This leads to the notion of the *Lebesgue integral*, which we shall spend this week and the next constructing. The Lebesgue integral can handle a very large class of functions, including all the Riemann integrable functions but also many others as well; in fact, it is safe to say that it can integrate virtually any function that one actually needs in mathematics, at least if one works on Euclidean spaces and everything is absolutely integrable. (If one assumes the axiom of choice, then there are still some pathological functions one can construct which cannot be integrated by the Lebesgue integral, but these functions will not come up in real-life applications).

- Before we turn to the details, we begin with an informal discussion. In order to understand how to compute an integral $\int_\Omega f$, we must first understand a more basic and fundamental question: how does one compute the *length/area/volume* of Ω? To see why this question is connected to that of integration, observe that if one integrates the function 1 on the set Ω, then one should obtain the length of Ω (if Ω is one-dimensional), the area of Ω (if Ω is two-dimensional), or the volume of Ω (if Ω is three-dimensional). To avoid splitting into cases depending on the dimension, we shall refer to the *measure* of Ω as either the length, area, volume, (or hypervolume, etc.) of Ω, depending on what Euclidean space $\mathbf{R}^n$ we are working in.

- Ideally, to every subset Ω of $\mathbf{R}^n$ we would like to associate a non-negative number $m(\Omega)$, which will be the measure of Ω (i.e. the length, area, volume, etc.). We allow the possibility for $m(\Omega)$ to be zero (e.g. if Ω is just a single point or the empty set) or for $m(\Omega)$

to be infinite (e.g. if Ω is all of $\mathbf{R}^n$). This measure should obey certain reasonable properties; for instance, the measure of the unit cube $(0,1)^n := \{(x_1, \ldots, x_n) : 0 < x_i < 1\}$ should equal 1, we should have $m(A \cup B) = m(A) + m(B)$ if A and B are disjoint, we should have $m(A) \leq m(B)$ whenever $A \subseteq B$, and we should have $m(x+A) = m(A)$ for any $x \in \mathbf{R}^n$ (i.e. if we shift A by the vector x the measure should be the same).

- Rather amazingly, it turns out that such a measure is impossible; one cannot assign a non-negative number to *every* subset of $\mathbf{R}^n$ which has the above properties. This is quite a surprising fact, as it goes against one's intuitive concept of volume; we shall prove it later in these notes. (An even more dramatic example of this failure of intuition is the *Banach-Tarski paradox*, in which a unit ball in $\mathbf{R}^3$ is decomposed into five pieces, and then the five pieces are reassembled via translations and rotations to form two complete and disjoint unit balls, thus violating any concept of conservation of volume; however we will not discuss this paradox here).

- What these paradoxes mean is that it is impossible to find a reasonable way to assign a measure to every single subset of $\mathbf{R}^n$. However, we can salvage matters by only measuring a certain class of sets in $\mathbf{R}^n$ - the *measurable sets*. These are the only sets Ω for which we will define the measure $m(\Omega)$, and once one restricts one's attention to measurable sets, one recovers all the above properties again. Furthermore, almost all the sets one encounters in real life are measurable (e.g. all open and closed sets will be measurable), and so this turns out to be good enough to do analysis.

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The goal: Lebesgue measure

- Let $\mathbf{R}^n$ be a Euclidean space. Our goal in this week's notes is to define a concept of *measurable set*, which will be a special kind of subset of $\mathbf{R}^n$, and for every such measurable set $\Omega \subset \mathbf{R}^n$, we will define the *Lebesgue measure* $m(\Omega)$ to be a certain number in $[0, \infty]$. The concept of measurable set will obey the following properties:

- (i) (Borel property) Every open set in $\mathbf{R}^n$ is measurable, as is every closed set.
- (ii) (Complementarity) If Ω is measurable, then $\mathbf{R}^n \backslash \Omega$ is also measurable.
- (iii) (Boolean algebra property) If $(\Omega_j)_{j \in J}$ is any finite collection of measurable sets (so J is finite), then the union $\bigcup_{j \in J} \Omega_j$ and intersection $\bigcap_{j \in J} \Omega_j$ are also measurable.
- (iv) (σ-algebra property) If $(\Omega_j)_{j \in J}$ are any countable collection of measurable sets (so J is countable), then the union $\bigcup_{j \in J} \Omega_j$ and intersection $\bigcap_{j \in J} \Omega_j$ are also measurable.
- Note that some of these properties are redundant; for instance, (iv) will imply (iii), and once one knows all open sets are measurable, (ii) will imply that all closed sets are measurable also. The properties (i-iv) will ensure that virtually every set one cares about is measurable; though as indicated in the introduction, there do exist non-measurable sets.
- To every measurable set Ω, we associate the *Lebesgue measure* $m(\Omega)$ of Ω, which will obey the following properties:
- (v) (Empty set) The empty set $\emptyset$ has measure $m(\emptyset) = 0$.
- (vi) (Positivity) We have $0 \leq m(\Omega) \leq +\infty$ for every measurable set Ω.
- (vii) (Monotonicity) If $A \subseteq B$, and A and B are both measurable, then $m(A) \leq m(B)$.
- (viii) (Finite sub-additivity) If $(A_j)_{j \in J}$ are a finite collection of measurable sets (so J is finite), then $m(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m(A_j)$.
- (ix) (Finite additivity) If $(A_j)_{j \in J}$ are a finite collection of *disjoint* measurable sets, then $m(\bigcup_{j \in J} A_j) = \sum_{j \in J} m(A_j)$.
- (x) (Countable sub-additivity) If $(A_j)_{j \in J}$ are a countable collection of measurable sets (so J is countable), then $m(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m(A_j)$.
- (xi) (Countable additivity) If $(A_j)_{j \in J}$ are a countable collection of *disjoint* measurable sets, then $m(\bigcup_{j \in J} A_j) = \sum_{j \in J} m(A_j)$.

- (xii) (Normalization) The unit cube $[0,1]^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : 0 \leq x_j \leq n$ for all $1 \leq j \leq n\}$ has measure $m([0,1]^n) = 1$.

- (xiii) (Translation invariance) If Ω is a measurable set, and $x \in \mathbf{R}^n$, then $x + \Omega := \{x + y : y \in \Omega\}$ is also measurable, and $m(x + \Omega) = m(\Omega)$.

- Again, many of these properties are redundant; for instance the countable additivity property can be used to deduce the finite additivity property, which in turn can be used to derive monotonicity (when combined with the positivity property). One can also obtain the sub-additivity properties from the additivity ones. Note that $m(\Omega)$ can be $+\infty$, and so in particular some of the sums in the above properties may also equal $+\infty$. (Since everything is positive we will never have to deal with indeterminate forms such as $-\infty + +\infty$).

- Our goal for this week can then be stated thus:

- **Existence of Lebesgue measure**. There exists a concept of a measurable set, and a way to assign a number $m(\Omega)$ to every measurable subset $\Omega \subseteq \mathbf{R}^n$, which obeys all of the properties (i)-(xiii).

- It turns out that Lebesgue measure is pretty much unique; any other concept of measurability and measure which obeys axioms (i)-(xiii) will largely co-incide with the construction we give. However there are other measures which obey only some of the above axioms. This leads to *measure theory*, which is an entire subject in itself and will not be pursued here; however we do remark that the concept of measures is very important in modern probability, and in the finer points of analysis (e.g. in the theory of distributions).

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First attempt: Outer measure

- We begin by using a rather naive approach to finding the measure of a set - try to cover it by boxes, and then add up the volume of each box. This approach will almost work, giving us a concept called *outer measure* which can be applied to every set and obeys all of the properties (v)-(xiii) except for the additivity properties (ix), (xi). Later we will have to modify outer measure slightly to recover the additivity property.

- We begin by starting with the notion of an open box.

- **Definition** A (open) *box* B in $\mathbf{R}^n$ is any set of the form

$$B = \prod_{i=1}^{n}(a_i, b_i) := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in (a_i, b_i) \text{ for all } 1 \leq i \leq n\},$$

where $b_i \geq a_i$ are real numbers. We define the *volume* $vol(B)$ of this box to be the number

$$vol(B) := \prod_{i=1}^{n}(b_i - a_i) = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n).$$

- For instance, the unit cube $(0,1)^n$ is a box, and has volume 1. In one dimension $n = 1$, boxes are the same as open intervals. One can easily check that in general dimension that open boxes are indeed open. Note that if we have $b_i = a_i$ for some i, then the box becomes empty, and has volume 0, but we still consider this to be a box (albeit a rather silly one). Sometimes we will use $vol_n(B)$ instead of $vol(B)$ to emphasize that we are dealing with n-dimensional volume, thus for instance $vol_1(B)$ would be the length of a one-dimensional box B, $vol_2(B)$ would be the area of a two-dimensional box B, etc.

- We of course expect the measure $m(B)$ of a box to be the same as the volume $vol(B)$ of that box. This is in fact an inevitable consequence of the axioms (i)-(xiii) (see Homework).

- **Definition** Let $\Omega \subseteq \mathbf{R}^n$ be a subset of $\mathbf{R}^n$. We say that a collection $(B_j)_{j \in J}$ of boxes *cover* Ω iff $\Omega \subseteq \bigcup_{j \in J} B_j$.

- Suppose $\Omega \subseteq \mathbf{R}^n$ can be covered by a finite or countable collection of boxes $(B_j)_{j \in J}$. If we wish Ω to be measurable, and if we wish to have a measure obeying the monotonicity and sub-additivity properties (vii), (viii), (x) and if we wish $m(B_j) = vol(B_j)$ for every box j, then we must have

$$m(\Omega) \leq m(\bigcup_{j \in J} B_j) \leq \sum_{j \in J} m(B_j) = \sum_{j \in J} vol(B_j).$$

$$m(\Omega) \leq \inf\{\sum_{j=1}^{\infty} vol(B_j) : (B_j)_{j \in J} \text{ is a finite or countable cover of } \Omega \text{ by boxes}\}.$$

- Inspired by this, we define

- **Definition** If Ω is a set, we define the *outer measure* $m^*(\Omega)$ of Ω to be the quantity

$$m^*(\Omega) := \inf\{\sum_{j=1}^{\infty} vol(B_j) : (B_j)_{j \in J} \text{ is a finite or countable cover of } \Omega \text{ by boxes}\}.$$

- Since $\sum_{j=1}^{\infty} vol(B_j)$ is non-negative, we know that $m^*(\Omega) \geq 0$ for all Ω. However, it is quite possible that $m^*(\Omega)$ could equal $+\infty$. Note that because we are allowing ourselves to use a countable number of boxes, that every subset of $\mathbf{R}^n$ has at least one countable cover by boxes; in fact $\mathbf{R}^n$ itself can be covered by countably many translates of the unit cube $(0,1)^n$ (how?). We will sometimes write $m_n^*(\Omega)$ instead of $m^*(\Omega)$ to emphasize the fact that we are using n-dimensional outer measure.

- Note that outer measure can be defined for every single set (not just the measurable ones), because we can take the infimum of any non-empty set. As we just mentioned, the positivity property (vi) is also obvious. Several other desirable properties are also easy to verify:

- **Lemma 1.** Outer measure has the following five properties:

- (v) (Empty set) The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.

- (vii) (Monotonicity) If $A \subseteq B \subseteq \mathbf{R}^n$, then $m^*(A) \leq m^*(B)$.

- (viii) (Finite sub-additivity) If $(A_j)_{j \in J}$ are a finite collection of subsets of $\mathbf{R}^n$, then $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$.

- (x) (Countable sub-additivity) If $(A_j)_{j \in J}$ are a countable collection of subsets of $\mathbf{R}^n$, then $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$.

- (xiii) (Translation invariance) If Ω is a subset of $\mathbf{R}^n$, and $x \in \mathbf{R}^n$, then $m^*(x + \Omega) = m^*(\Omega)$.

- **Proof.** See Week 8 homework.
- The outer measure of a closed box is also what we expect:
- **Proposition 2.** For any closed box

  $$B = \prod_{i=1}^{n} [a_i, b_i] := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in [a_i, b_i] \text{ for all } 1 \leq i \leq n\},$$

  we have

  $$m^*(B) = \prod_{i=1}^{n} (b_i - a_i).$$

- **Proof.** Clearly, we can cover the closed box $B = \prod_{i=1}^{n} [a_i, b_i]$ by the open box $\prod_{i=1}^{n} (a_i - \varepsilon, b_i + \varepsilon)$ for every $\varepsilon > 0$. Thus we have

  $$m^*(B) \leq vol(\prod_{i=1}^{n} (a_i - \varepsilon, b_i + \varepsilon)) = \prod_{i=1}^{n} (b_i - a_i + 2\varepsilon)$$

  for every $\varepsilon > 0$. Taking limits as $\varepsilon \to 0$, we obtain

  $$m^*(B) \leq \prod_{i=1}^{n} (b_i - a_i).$$

  To finish the proof, we need to show that

  $$m^*(B) \geq \prod_{i=1}^{n} (b_i - a_i).$$

  By the definition of $m^*(B)$, it suffices to show that

  $$\sum_{j \in J} vol(B_j) \geq \prod_{i=1}^{n} (b_i - a_i)$$

  whenever $(B_j)_{j \in J}$ is a finite or countable cover of B.

- Since B is closed and bounded, it is compact (by the Heine-Borel theorem, see Week 2 notes), and in particular every open cover has a finite subcover (Theorem 9 of Week 2 notes). Thus to prove the above inequality for countable covers, it suffices to do it for finite covers (since if $(B_j)_{j \in J'}$ is a finite subcover of $(B_j)_{j \in J}$ then $\sum_{j \in J} vol(B_j)$ will be greater than or equal to $\sum_{j \in J'} vol(B_j)$).

- To summarize, our goal is now to prove that

$$\sum_{j \in J} vol(B^{(j)}) \geq \prod_{i=1}^{n} (b_i - a_i) \qquad (*)$$

whenever $(B^{(j)})_{j \in J}$ is a finite cover of $\prod_{j=1}^{n} [a_i, b_i]$; we have changed the subscript B_j to superscript $B^{(j)}$ because we will need the subscripts to denote components.

- To prove the above inequality (*), we shall use induction on the dimension n. First we consider the base case $n = 1$. Here B is just a closed interval $B = [a, b]$, and each box $B^{(j)}$ is just an open interval $B^{(j)} = (a_j, b_j)$. We have to show that

$$\sum_{j \in J} (b_j - a_j) \geq (b - a).$$

To do this we use the Riemann integral. For each $j \in J$, let $f^{(j)} : \mathbf{R} \to \mathbf{R}$ be the function such that $f^{(j)}(x) = 1$ when $x \in (a_j, b_j)$ and $f^{(j)}(x) = 0$ otherwise. Then we have that $f^{(j)}$ is Riemann integrable (because it is piecewise constant, and compactly supported) and

$$\int_{-\infty}^{\infty} f^{(j)} = b_j - a_j.$$

Summing this over all $j \in J$, and interchanging the integral with the finite sum, we have

$$\int_{-\infty}^{\infty} \sum_{j \in J} f^{(j)} = \sum_{j \in J} b_j - a_j.$$

But since the intervals (a_j, b_j) cover $[a, b]$, we have $\sum_{j \in J} f^{(j)}(x) \geq 1$ for all $x \in [a, b]$ (why?). For all other values if x, we have $\sum_{j \in J} f^{(j)}(x) \geq 0$. Thus

$$\int_{-\infty}^{\infty} \sum_{j \in J} f^{(j)} \geq \int_{[a,b]} 1 = b - a$$

and the claim follows by combining this inequality with the previous equality. This proves (*) when $n = 1$.

- Now assume inductively that $n > 1$, and we have already proven the inequality (*) for dimensions $n-1$. We shall use a similar argument to the preceding one.

- Each box $B^{(j)}$ is now of the form

$$B^{(j)} = \prod_{i=1}^{n} (a_i^{(j)}, b_i^{(j)}).$$

We can write this as

$$B^{(j)} = A^{(j)} \times (a_n^{(j)}, b_n^{(j)})$$

where $A^{(j)}$ is the $n-1$-dimensional box $A^{(j)} := \prod_{i=1}^{n-1} (a_i^{(j)}, b_i^{(j)})$. Note that

$$vol(B^{(j)}) = vol_{n-1}(A^{(j)})(b_n^{(j)} - a_n^{(j)})$$

where we have subscripted vol_{n-1} by $n-1$ to emphasize that this is $n-1$-dimensional volume being referred to here. We similarly write

$$B = A \times [a_n, b_n]$$

where $A := \prod_{i=1}^{n-1} [a_i, b_i]$, and again note that

$$vol(B) = vol_{n-1}(A)(b_n - a_n).$$

For each $j \in J$, let $f^{(j)}$ be the function such that $f^{(j)}(x_n) = vol_{n-1}(A^{(j)})$ for all $x_n \in (a_n^{(j)}, b_n^{(j)})$, and $f^{(j)}(x_n) = 0$ for all other x_n. Then $f^{(j)}$ is Riemann integrable and

$$\int_{-\infty}^{\infty} f^{(j)} = vol_{n-1}(A^{(j)})(b_n^{(j)} - a_n^{(j)}) = vol(B^{(j)})$$

and hence

$$\sum_{j \in J} vol(B^{(j)}) = \int_{-\infty}^{\infty} \sum_{j \in J} f^{(j)}.$$

Now let $x_n \in [a_n, b_n]$ and $(x_1, \ldots, x_{n-1}) \in A$. Then $(x_1, \ldots, x_n)$ lies in B, and hence lies in one of the $B^{(j)}$. Clearly we have $x_n \in (a_n^{(j)}, b_n^{(j)})$, and $(x_1, \ldots, x_{n-1}) \in A^{(j)}$. In particular, we see that for each $x_n \in [a_n, b_n]$, the set

$$\{A^{(j)} : j \in J; x_n \in (a_n^{(j)}, b_n^{(j)})\}$$

of $n-1$-dimensional boxes covers A. Applying the inductive hypothesis (*) we thus see that

$$\sum_{j \in J : x_n \in (a_n^{(j)}, b_n^{(j)})} vol_{n-1}(A^{(j)}) \geq vol_{n-1}(A),$$

or in other words

$$\sum_{j \in J} f^{(j)}(x_n) \geq vol_{n-1}(A).$$

Integrating this over $[a_n, b_n]$, we obtain

$$\int_{[a_n,b_n]} \sum_{j \in J} f^{(j)} \geq vol_{n-1}(A)(b_n - a_n) = vol(B)$$

and in particular

$$\int_{-\infty}^{\infty} \sum_{j \in J} f^{(j)} \geq vol_{n-1}(A)(b_n - a_n) = vol(B)$$

since $\sum_{j \in J} f^{(j)}$ is always non-negative. Combining this with our previous identity for $\int_{-\infty}^{\infty} \sum_{j \in J} f^{(j)}$ we obtain (*), and the induction is complete. □

- Once we obtain the measure of a closed box, the corresponding result for an open box is easy:

- **Corollary 3.** For any open box

$$B = \prod_{i=1}^{n} (a_i, b_i) := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in [a_i, b_i] \text{ for all } 1 \leq i \leq n\},$$

we have

$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i).$$

In particular, outer measure obeys the normalization (xii).

- **Proof.** We may assume that $b_i > a_i$ for all i, since if $b_i = a_i$ this follows from Lemma 1(v). Now observe that

$$\prod_{i=1}^{n}[a_i + \varepsilon, b_i - \varepsilon] \subset \prod_{i=1}^{n}(a_i, b_i) \subset \prod_{i=1}^{n}[a_i, b_i]$$

for all $\varepsilon > 0$, assuming that ε is small enough that $b_i - \varepsilon > a_i + \varepsilon$ for all i. Applying Proposition 2 and Lemma 1(vii) we obtain

$$\prod_{i=1}^{n}(b_i - a_i - 2\varepsilon) \leq m^*(\prod_{i=1}^{n}(a_i, b_i)) \leq \prod_{i=1}^{n}(b_i - a_i).$$

Sending $\varepsilon \to 0$ and using the squeeze test, one obtains the result. $\square$

- We now compute some examples of outer measure on the real line **R**.

- **Example.** Let us compute the one-dimensional measure of **R**. Since $(-R, R) \subset \mathbf{R}$ for all $R > 0$, we have

$$m^*(\mathbf{R}) \geq m^*((-R, R)) = 2R$$

by Corollary 3. Letting $R \to +\infty$ we thus see that $m^*(\mathbf{R}) = +\infty$.

- **Example.** Now let us compute the one-dimensional measure of **Q**. From Proposition 2 we see that for each rational number **Q**, the point $\{q\}$ has outer measure $m^*(\{q\}) = 0$. Since **Q** is clearly the union $\mathbf{Q} = \bigcup_{q \in \mathbf{Q}}\{q\}$ of all these rational points q, and **Q** is countable, we have

$$m^*(\mathbf{Q}) \leq \sum_{q \in \mathbf{Q}} m^*(\mathbf{Q}) = \sum_{q \in \mathbf{Q}} 0 = 0,$$

and so $m^*(Q)$ must equal zero. In fact, the same argument shows that every countable set has measure zero. (This, incidentally, gives another proof that the real numbers are uncountable).

- Incidentally, one consequence of the fact that $m^*(\mathbf{Q}) = 0$ is that given any $\varepsilon > 0$, it is possible to cover the rationals **Q** by a countable number of intervals whose total length is less than ε. This fact is somewhat unintuitive; can you see how to cover the rationals in this manner?

- **Example.** Now let us compute the one-dimensional measure of the irrationals $\mathbf{R}\backslash\mathbf{Q}$. From finite sub-additivity we have

$$m^*(\mathbf{R}) \leq m^*(\mathbf{R}\backslash\mathbf{Q}) + m^*(\mathbf{Q}).$$

  Since $\mathbf{Q}$ has outer measure 0, and $m^*(\mathbf{R})$ has outer measure $+\infty$, we thus see that the irrationals $\mathbf{R}\backslash\mathbf{Q}$ have outer measure $+\infty$. A similar argument shows that $[0,1]\backslash\mathbf{Q}$, the irrationals in $[0,1]$, have measure 1 (why?).

- **Example.** The unit interval $[0,1]$ in $\mathbf{R}$ has one-dimensional outer measure 1, by Proposition 2. But the interval $\{(x,0) : 0 \leq x \leq 1\}$ in $\mathbf{R}^2$ has two-dimensional outer measure 0 (why? Use Proposition 2 again). Thus one-dimensional outer measure and two-dimensional outer measure are quite different. Note that the above remarks and countable additivity imply that the entire x-axis of $\mathbf{R}^2$ has two-dimensional outer measure 0, despite the fact that $\mathbf{R}$ has infinite one-dimensional measure.

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Outer measure is not additive

- It would seem now that all we need to do is verify the additivity properties (ix), (xi), and we have everything we need to have a usable measure. Unfortunately, these properties fail for outer measure, even in one dimension $\mathbf{R}$.

- **Proposition 4 (Failure of countable additivity).** There exists a countable collection $(A_j)_{j\in J}$ of disjoint subsets of $\mathbf{R}$, such that $m^*(\bigcup_{j\in J} A_j) \neq \sum_{j\in J} m^*(A_j)$.

- **Proof.** (Optional) We shall need some notation. Let $\mathbf{Q}$ be the rationals, and $\mathbf{R}$ be the reals. We say that a set $A \subset \mathbf{R}$ is a *coset* of $\mathbf{Q}$ if it is of the form $A = x + \mathbf{Q}$ for some real number x. For instance, $\sqrt{2} + \mathbf{Q}$ is a coset of $\mathbf{R}$, as is $\mathbf{Q}$ itself, since $\mathbf{Q} = 0 + \mathbf{Q}$. Note that a coset A can correspond to several values of x; for instance $2 + \mathbf{Q}$ is exactly the same coset as $0 + \mathbf{Q}$. Also observe that it is not possible for two cosets to partially overlap; if $x + \mathbf{Q}$ intersects $y + \mathbf{Q}$ in even just a single point z, then $x - y$ must be rational (why? use the identity

$x - y = (x - z) - (y - z))$, and thus $x + \mathbf{Q}$ and $y + \mathbf{Q}$ must be equal (why?). So any two cosets are either identical or distinct.

- We observe that every coset A of the rationals $\mathbf{Q}$ has a non-empty intersection with $[0,1]$. Indeed, if A is a coset, then $A = x + \mathbf{Q}$ for some real number x. If we then pick a rational number q in $[-x, 1-x]$ then we see that $x + q \in [0,1]$, and thus $A \cap [0,1]$ contains $x + q$.

- Let $\mathbf{R}/\mathbf{Q}$ denote the set of all cosets of $\mathbf{Q}$; note that this is a set whose elements are themselves sets (of real numbers). For each coset A in $\mathbf{R}/\mathbf{Q}$, let us pick an element x_A of $A \cap [0,1]$. (This requires us to make an infinite number of choices, and thus requires the axiom of choice). Let E be the set of all such x_A, i.e. $E := \{x_A : A \in \mathbf{R}/\mathbf{Q}\}$. Note that $E \subseteq [0,1]$ by constrution.

- Now consider the set

$$X = \bigcup_{q \in \mathbf{Q} \cap [-1,1]} (q + E).$$

  Clearly this set is contained in $[-1,2]$ (since $q + x \in [-1,2]$ whenever $q \in [-1,1]$ and $x \in E \subseteq [0,1]$). We claim that this set contains the interval $[0,1]$. Indeed, for any $y \in [0,1]$, we know that y must belong to some coset A (for instance, it belongs to the coset $y + \mathbf{Q}$). But we also have x_A belonging to the same coset, and thus $y - x_A$ is equal to some rational q. Since y and x_A both live in $[0,1]$, then q lives in $[-1,1]$. Since $y = q + x_A$, we have $y \in q + E$, and hence $y \in X$ as desired.

- We claim that

$$m^*(X) \neq \sum_{q \in \mathbf{Q} \cap [-1,1]} m^*(q + E),$$

  which would prove the claim. To see why this is true, observe that since $[0,1] \subseteq X \subseteq [-1,2]$, that we have $1 \leq m^*(X) \leq 3$ by monotonicity and Proposition 2. For the right hand side, observe from translation invariance that

$$\sum_{q \in \mathbf{Q} \cap [-1,1]} m^*(q + E) = \sum_{q \in \mathbf{Q} \cap [-1,1]} m^*(E).$$

The set $\mathbf{Q} \cap [-1, 1]$ is countably infinite (why?). Thus the right-hand side is either 0 (if $m^*(E) = 0$) or $+\infty$ (if $m^*(E) > 0$). Either way, it cannot be between 1 and 3, and the claim follows. $\square$

- The above proof used the axiom of choice. This turns out to be absolutely necessary; one can prove using some advanced techniques in mathematical logic that if one does not assume the axiom of choice, then it is possible to have a mathematical model where outer measure is countably additive.

- One can refine the above argument, and show in fact that m^* is not finitely additive either:

- **Proposition 5 (Failure of finite additivity).** There exists a finite collection $(A_j)_{j \in J}$ of disjoint subsets of $\mathbf{R}$, such that $m^*(\bigcup_{j \in J} A_j) \neq \sum_{j \in J} m^*(A_j)$.

- **Proof.** This is accomplished by an indirect argument. Suppose for contradiction that m^* *was* finitely additive. Let E and X be the sets introduced in Proposition 4. From countable sub-additivity and translation invariance we have $m^*(X) \leq \sum_{q \in \mathbf{Q} \cap [-1,1]} m^*(q + E) = \sum_{q \in \mathbf{Q} \cap [-1,1]} m^*(E)$. Since we know that $1 \leq m^*(X) \leq 3$, we thus have $m^*(E) \neq 0$, since otherwise we would have $m^*(X) \leq 0$, contradiction.

- Since $m^*(E) \neq 0$, there exists a finite integer $n > 0$ such that $m^*(E) > 1/n$. Now let J be a finite subset of $\mathbf{Q} \cap [-1, 1]$ of cardinality 3n. If m^* were finitely additive, then we would have

$$m^*(\sum_{q \in J} q + E) = \sum_{q \in J} m^*(q + E) = \sum_{q \in J} m^*(E) > 3n\frac{1}{n} = 3.$$

But we know that $\sum_{q \in J} q + E$ is a subset of X, which has outer measure at most 3. This contradicts monotonicity. Hence m^* cannot be finitely additive. $\square$

* * * * *

Measurable sets

- In the previous section we saw that certain sets were badly behaved with respect to outer measure, in particular they could be used to contradict finite or countable additivity. However, those sets were rather pathological, being constructed using the axiom of choice and looking rather artificial. One would hope to be able to exclude them and then somehow recover finite and countable additivity. Fortunately, this can be done.

- **Definition** Let E be a subset of $\mathbf{R}^n$. We say that E is *Lebesgue measurable*, or *measurable* for short, iff we have the identity

  $$m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$$

  for every subset A of $\mathbf{R}^n$. If E is measurable, we define the *Lebesgue measure* of E to be $m(E) = m^*(E)$; if E is not measurable, we leave $m(E)$ undefined.

- In other words, E being measurable means that if we use the set E to divide up an arbitrary set A into two parts, we keep the additivity property. Of course, if m^* were finitely additive then every set E would be measurable; but we know from Proposition 5 that not every set is finitely additive. One can think of the measurable sets as the sets for which finite additivity works. We sometimes subscript $m(E)$ as $m_n(E)$ to emphasize the fact that we are using n-dimensional Lebesgue measure.

- We now begin showing that a large number of sets are indeed measurable. The empty set $E = \emptyset$ and the whole space $E = \mathbf{R}^n$ are clearly measurable. Here is another example of a measurable set:

- **Lemma 6.** The half-plane $\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}$ is measurable.

- **Proof.** See Week 8 homework. $\square$

- A similar argument to Lemma 6 shows that any half-plane of the form $\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_j > 0\}$ or $\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_j < 0\}$ is measurable.

- Now for some more properties of measurable sets.

- **Lemma 7**
- (a) If E is measurable, then $\mathbf{R}^n \backslash E$ is also measurable.
- (b) (Translation invariance) If E is measurable, and $x \in \mathbf{R}^n$, then $x+E$ is also measurable, and $m(x+E) = m(E)$.
- (c) If E_1 and E_2 are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable.
- (d) (Boolean algebra property) If $E_1, E_2, \ldots, E_N$ are measurable, then $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ are measurable.
- (e) Every open box, and every closed box, is measurable.
- **Proof.** See Week 8 homework. □
- From Lemma 7, we have proven properties (ii), (iii), (xiii) on our wish list of measurable sets, and we are making progress towards (i). We also have finite additivity (property (ix) on our wish list):
- **Lemma 8. (Finite additivity)** If $(E_j)_{j \in J}$ are a finite collection of *disjoint* measurable sets and any set A (not necessarily measurable), we have
$$m^*(A \cap \bigcup_{j \in J} E_j) = \sum_{j \in J} m^*(A \cap E_j).$$
Furthermore, we have $m(\bigcup_{j \in J} E_j) = \sum_{j \in J} m(E_j)$.
- **Proof.** See Week 8 homework. □
- Note that Lemma 8 and Proposition 5 imply that there exist non-measurable sets. (Indeed, one can show that the set E used in Propositions 4 and 5 are non-measurable).
- **Corollary 9** If $A \subseteq B$ are two measurable sets, then $B \backslash A$ is also measurable, and
$$m(B \backslash A) = m(B) - m(A).$$
- **Proof.** See Week 8 homework. □
- Now we show countable additivity.

- **Lemma 10. (Countable additivity)** If $(E_j)_{j\in J}$ are a countable collection of *disjoint* measurable sets, then $\bigcup_{j\in J} E_j$ is measurable, and $m(\bigcup_{j\in J} E_j) = \sum_{j\in J} m(E_j)$.

- **Proof.** Let $E := \bigcup_{j\in J} E_j$. Our first task will be to show that E is measurable. Thus, let A be any other measurable set; we need to show that
$$m^*(A) = m^*(A\cap E) + m^*(A\backslash E).$$

- Since J is countable, we may write $J = \{j_1, j_2, j_3, \ldots\}$. Note that
$$A \cap E = \bigcup_{k=1}^{\infty} (A \cap E_{j_k})$$
(why?) and hence by countable sub-additivity
$$m^*(A\cap E) \leq \sum_{k=1}^{\infty} m^*(A\cap E_{j_k}).$$
We rewrite this as
$$m^*(A\cap E) \leq \sup_{N\geq 1} \sum_{k=1}^{N} m^*(A\cap E_{j_k}).$$
Let F_N be the set $F_N := \bigcup_{k=1}^{N} E_{j_k}$. Since the $A\cap E_{j_k}$ are all disjoint, and their union is $A\cap F_N$, we see from Lemma 8 that
$$\sum_{k=1}^{N} m^*(A\cap E_{j_k}) = m^*(A\cap F_N)$$
and hence
$$m^*(A\cap E) \leq \sup_{N\geq 1} m^*(A\cap F_N).$$
Now we look at $A\backslash E$. Since $F_N \subseteq E$ (why?), we have $A\backslash E \subseteq A\backslash F_N$ (why?). By monotonicity, we thus have
$$m^*(A\backslash E) \leq m^*(A\backslash F_N)$$

for all N. In particular, we see that

$$m^*(A \cap E) + m^*(A \backslash E) \leq \sup_{N \geq 1} m^*(A \cap F_N) + m^*(A \backslash E)$$

$$\leq \sup_{N \geq 1} m^*(A \cap F_N) + m^*(A \backslash F_N).$$

But from Lemma 8 we know that F_N is measurable, and hence

$$m^*(A \cap F_N) + m^*(A \backslash F_N) = m^*(A).$$

Putting this all together we obtain

$$m^*(A \cap E) + m^*(A \backslash E) \leq m^*(A).$$

But from finite sub-additivity we have

$$m^*(A \cap E) + m^*(A \backslash E) \geq m^*(A)$$

and the claim follows. This shows that E is measurable.

- To finish the lemma, we need to show that $m(E) = \sum_{j \in J} m(E_j)$. We first observe from countable sub-additivity that

$$m(E) \leq \sum_{j \in J} m(E_j) = \sum_{k=1}^{\infty} m(E_{j_k}).$$

On the other hand, by finite additivity and monotonicity we have

$$m(E) \geq m(F_N) = \sum_{k=1}^{N} m(E_{j_k}).$$

Taking limits as $N \to \infty$ we obtain

$$m(E) \geq \sum_{k=1}^{\infty} m(E_{j_k})$$

and thus we have

$$m(E) = \sum_{k=1}^{\infty} m(E_{j_k}) = \sum_{j \in J} m(E_j)$$

as desired. $\square$

- This proves property (xi) on our wish list. Next, we do countable unions and intersections.

- **Lemma 11.** **(σ-algebra property)** If $(\Omega_j)_{j \in J}$ are any countable collection of measurable sets (so J is countable), then the union $\bigcup_{j \in J} \Omega_j$ and the intersection $\bigcap_{j \in J} \Omega_j$ are also measurable.

- **Proof.** See Week 8 homework. $\square$

- The final property left to verify on our wish list is (i). We first need a preliminary lemma.

- **Lemma 12.** Every open set can be written as a countable or finite union of open boxes.

- **Proof.** We first need some notation. Call a box $B = \prod_{i=1}^n (a_i, b_i)$ *rational* if all of its components a_i, b_i are rational numbers. Observe that there are only a countable number of rational boxes (this is since a rational box is described by $2n$ rational numbers, and so has the same cardinality as $\mathbf{Q}^{2n}$. But $\mathbf{Q}$ is countable, and the Cartesian product of any finite number of countable sets is countable).

- We make the following claim: given any open ball $B(x, r)$, there exists a rational box B which is contained in $B(x, r)$ and which contains x. To prove this claim, write $x = (x_1, \ldots, x_n)$. For each $1 \leq i \leq n$, let a_i and b_i be rational numbers such that

$$x_i - \frac{r}{n} < a_i < x_i < b_i < x_i + \frac{r}{n}.$$

  Then it is clear that the box $\prod_{i=1}^n (a_i, b_i)$ is rational and contains x. A simple computation using Pythagoras's theorem (or the triangle inequality) also shows that this box is contained in $B(x, r)$; we leave this to the reader.

- Now let E be an open set, and let Σ be the set of all rational boxes B which are subsets of E, and consider the union $\bigcup_{B \in \Sigma} B$ of all those boxes. Clearly, this union is contained in E, since every box in Σ is contained in E by construction. On the other hand, since E is open, we see that for every $x \in E$ there is a ball $B(x, r)$ contained in E, and by the previous claim this ball contains a rational box which contains x. In particular, x is contained in $\bigcup_{B \in \Sigma} B$. Thus we have

$$E = \bigcup_{B \in \Sigma} B$$

as desired; note that Σ is countable or finite because it is a subset of the set of all rational boxes, which is countable. $\square$

- **Lemma 13. (Borel property)** Every open set, and every closed set, is Lebesgue measurable.

- **Proof.** It suffices to do this for open sets, since the claim for closed sets then follows by Lemma 7(a) (i.e. property (ii)).

- Let E be an open set. By Lemma 12, E is the countable union of boxes. Since we already know that boxes are measurable, and that the countable union of measurable sets is measurable, the claim follows. $\square$

- The construction of Lebesgue measure and its basic properties are now complete. Now we make the next step in constructing the Lebesgue integral - describing the class of functions we can integrate.

* * * * *

Measurable functions

- In the theory of the Riemann integral, we are only able to integrate a certain class of functions - the Riemann integrable functions. We will now be able to integrate a much larger range of functions - the *measurable functions.* (OK, to be precise, we can only handle those measurable functions which are absolutely integrable, but more on that later).

- **Definition** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. A function f is *measurable* iff $f^{-1}(V)$ is measurable for every open set $V \subseteq \mathbf{R}^m$.

- As discussed earlier, most sets that we deal with in real life are measurable, so it is only natural to learn that most functions we deal with in real life are also measurable. For instance, continuous functions are automatically measurable:

- **Lemma 14.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be continuous. Then f is also measurable.

- **Proof.** Let V be any open subset of $\mathbf{R}^m$. Then since f is continuous, $f^{-1}(V)$ is open relative to Ω (Theorem 13(c) of Week 2 notes), i.e. $f^{-1}(V) = W \cap \Omega$ for some open set $W \subseteq \mathbf{R}^n$ (Proposition 7(a) of Week 1 notes). Since W is open, it is measurable; since Ω is measurable, $W \cap \Omega$ is also measurable. $\square$

- Because of Lemma 12, we have an easy criterion to test whether a function is measurable or not:

- **Lemma 15.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. Then f is measurable if and only if $f^{-1}(B)$ is measurable for every open box B.

- **Proof.** See Week 9 homework. $\square$

- **Corollary 16.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. Suppose that $f = (f_1, \ldots, f_m)$, where $f_j : \Omega \to \mathbf{R}$ is the j^{th} co-ordinate of f. Then f is measurable if and only if all of the f_j are individually measurable.

- **Proof.** See Week 9 homework. $\square$

- Unfortunately, it is not true that the composition of two measurable functions is automatically measurable; however we can do the next best thing: a continuous function applied to a measurable function is measurable.

- **Lemma 17.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let W be an open subset of $\mathbf{R}^m$. If $f : \Omega \to W$ is measurable, and $g : W \to \mathbf{R}^p$ is continuous, then $g \circ f : \Omega \to \mathbf{R}^p$ is measurable.

- **Proof.** See Week 9 homework. $\square$

- This has an immediate corollary:

- **Corollary 18.** Let Ω be a measurable subset of $\mathbf{R}^n$. If $f : \Omega \to \mathbf{R}$ is a measurable function, then so is $|f|$, $\max(f, 0)$, and $\min(f, 0)$.

- **Proof.** Apply Lemma 17 with $g(x) := |x|$, $g(x) := \max(x, 0)$, and $g(x) := \min(x, 0)$. $\square$

- A slightly less immediate corollary:

- **Corollary 19.** Let Ω be a measurable subset of $\mathbf{R}^n$. If $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ are measurable functions, then so is $f + g$, $f - g$, fg, $\max(f, g)$, and $\min(f, g)$. If $g(x) \neq 0$ for all $x \in \Omega$, then f/g is also measurable.

- **Proof.** Consider $f + g$. We can write this as $k \circ h$, where $h : \Omega \to \mathbf{R}^2$ is the function $h(x) = (f(x), g(x))$, and $k : \mathbf{R}^2 \to \mathbf{R}$ is the function $k(a, b) := a + b$. Since f, g are measurable, then h is also measurable by Corollary 16. Since k is continuous, we thus see from Lemma 17 that $k \circ h$ is measurable, as desired. A similar argument deals with all the other cases; the only thing concerning the f/g case is that the space $\mathbf{R}^2$ must be replaced with $\{(a, b) \in \mathbf{R}^2 : b \neq 0\}$ in order to keep the map $(a, b) \mapsto a/b$ continuous and well-defined. $\square$

- Another characterization of measurable functions is given by

- **Lemma 20.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a function. Then f is measurable if and only if $f^{-1}((a, \infty))$ is measurable for every real number a.

- **Proof.** See Week 9 homework. $\square$

- Inspired by this Lemma, we extend the notion of a measurable function to the extended real number system $\mathbf{R}^* := \mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$:

- **Definition** Let Ω be a measurable subset of $\mathbf{R}^n$. A function $f : \Omega \to \mathbf{R}^*$ is said to be *measurable* iff we $f^{-1}((a, \infty))$ is measurable for every real number a.

- Thus the notion of measurability for functions taking values in the extended reals $\mathbf{R}^*$ is compatible with that for functions taking place in just the reals $\mathbf{R}$.

- Now we show that limits of sequences of measurable functions are also measurable.

- **Lemma 21.** Let Ω be a measurable subset of $\mathbf{R}^n$. For each positive integer n, let $f_n : \Omega \to \mathbf{R}^*$ be a measurable function. Then the functions $\sup_{n \geq 1} f_n$, $\inf_{n \geq 1} f_n$, $\limsup_{n \to \infty} f_n$, and $\liminf_{n \to \infty} f_n$ are also measurable. In particular, if the f_n converge pointwise to another function $f : \Omega \to \mathbf{R}$, then f is also measurable.

- **Proof.** We first prove the claim about $\sup_{n \geq 1} f_n$. Call this function g. We have to prove that $g^{-1}((a, \infty))$ is measurable for every a. But by the definition of supremum, we have

  $$g^{-1}((a, \infty)) = \bigcup_{n \geq 1} f_n^{-1}((a, \infty))$$

  (why?), and the claim follows since the countable union of measurable sets is again measurable.

- A similar argument works for $\inf_{n \geq 1} f_n$. The claim for lim sup and lim inf then follow from the identities

  $$\limsup_{n \to \infty} f_n = \inf_{N \geq 1} \sup_{n \geq N} f_n$$

  and

  $$\liminf_{n \to \infty} f_n = \sup_{N \geq 1} \inf_{n \geq N} f_n$$

  (see math 131A). □

- As you can see, just about anything one does to a measurable function will produce another measurable function. This is basically why almost every function one deals with in mathematics is measurable. (Indeed, the only way to construct non-measurable functions is via artificial means such as invoking the axiom of choice).

* * * * *

Simple functions

- One way to approach the theory of the Riemann integral is to begin by integrating a particularly simple class of functions, namely the *piecewise constant* functions; see for instance Week 9-10 of my Math 131AH notes. Among other things, piecewise constant functions only attain a finite number of values (as opposed to most functions in real life, which can take an infinite number of values). Once one learns how to integrate piecewise constant functions, one can then integrate other Riemann integrable functions by a similar procedure.

- A similar philosophy is used to construct the Lebesgue integral. We shall begin by considering a special subclass of measurable functions - the *simple* functions. Next week we will show how to integrate simple functions, and then from there we will integrate all measurable functions (or at least the absolutely integrable ones).

- **Definition** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a measurable function. We say that f is a *simple function* if the image $f(\Omega)$ is finite. In other words, there exists a finite number of real numbers $c_1, c_2, \ldots, c_N$ such that for every $x \in \Omega$, we have $f(x) = c_j$ for some $1 \leq j \leq N$.

- **Example.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let E be a measurable subset of Ω. We define the *characteristic function* $\chi_E : \Omega \to \mathbf{R}$ by setting $\chi_E(x) := 1$ if $x \in E$, and $\chi_E(x) := 0$ if $x \notin E$. (In some texts, χ_E is also written 1_E; Rudin uses K_E but this is rare nowadays). Then χ_E is a measurable function (why?), and is a simple function, because the image $\chi_E(\Omega)$ is $\{0, 1\}$ (or $\{0\}$ if E is empty, or $\{1\}$ if $E = \Omega$).

- We remark on three basic properties of simple functions. First, the space of simple functions forms a vector space:

- **Lemma 22.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be simple functions. Then $f + g$ is also a simple function. Also, for any scalar $c \in \mathbf{R}$, the function cf is also a simple function.

- **Proof.** See Week 9 homework. □

- Secondly, the space of simple functions is generated by the characteristic functions, i.e. every simple function is a linear combination of characteristic functions:

- **Lemma 23.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a simple function. Then there exists a finite number of real numbers $c_1, \ldots, c_N$, and a finite number of disjoint measurable sets $E_1, E_2, \ldots, E_N$ in Ω, such that $f = \sum_{i=1}^N c_i \chi_{E_i}$.

- **Proof.** See Week 9 homework. $\square$

- Thirdly, we can approximate general measurable functions by simple ones.

- **Lemma 24.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a measurable function. Suppose that f is always non-negative, i.e. $f(x) \geq 0$ for all $x \in \Omega$. Then there exists a sequence $f_1, f_2, f_3, \ldots$ of simple functions, $f_n : \Omega \to \mathbf{R}$, such that the f_n are non-negative and increasing,

$$0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots \text{ for all } x \in \Omega$$

and converge pointwise to f:

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in \Omega.$$

- **Proof.** See Week 9 homework. $\square$

- Next week, we will integrate these simple functions and then construct the Lebesgue integral.