Math 131BH - Week 2
Textbook pages covered: 36-40, 83-93

- Subsequences
- Cauchy sequences and complete metric spaces
- Compact sets
- Continuous functions
- Continuity and compactness
- Continuity and connectedness

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Subsequences

- We review the notion of a *subsequence* from Math 131AH. Suppose that $(x^{(n)})_{n=m}^\infty$ is a sequence of points in a metric space (X, d). Suppose that $n_1, n_2, n_3, \ldots$ is an increasing sequence of integers which are at least as large as m, thus
$$m \leq n_1 < n_2 < n_3 < \ldots.$$
Then we call the sequence $(x^{(n_j)})_{j=1}^\infty$ a *subsequence* of the original sequence $(x^{(n)})_{n=m}^\infty$.
- **Examples:** the sequence $((\frac{1}{j^2}, \frac{1}{j^2}))_{j=1}^\infty$ in $\mathbf{R}^2$ is a subsequence of the sequence $((\frac{1}{n}, \frac{1}{n}))_{n=1}^\infty$ (in this case, $n_j := j^2$). The sequence $1, 1, 1, 1, \ldots$ is a subsequence of $1, 0, 1, 0, 1, \ldots$.
- If a sequence converges, then so does all of its subsequences:
- **Lemma 1.** Let $(x^{(n)})_{n=m}^\infty$ be a sequence in (X, d) which converges to some limit x_0. Then every subsequence $(x^{(n_j)})_{j=1}^\infty$ of that sequence also converges to x_0.
- **Proof.** See Week 2 homework. □

- On the other hand, it is possible for a subsequence to be convergent without the sequence as a whole being convergent. For example, the sequence $1, 0, 1, 0, 1, \ldots$ is not convergent, even though certain subsequences of it (such as $1, 1, 1, \ldots$ converges).

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Cauchy sequences and complete metric spaces

- Next, we review the notion of a *Cauchy sequence* from Math 131AH. Informally speaking, a Cauchy sequence is a sequence which may or may not be converging to some final limit, but whose elements are definitely converging *to each other*. The formal definition is as follows:

- **Definition.** Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d). We say that this sequence is a *Cauchy sequence* iff for every $\varepsilon > 0$, there exists an $N \geq m$ such that $d(x^{(j)}, x^{(k)}) < \varepsilon$ for all $j, k \geq N$.

- This should agree with the definitions of Cauchy sequences you may have seen in other courses, such as Math 131AH. As for examples of Cauchy sequences, every convergent sequence is a Cauchy sequence:

- **Lemma 2.** Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then $(x^{(n)})_{n=m}^{\infty}$ is also a Cauchy sequence.

- **Proof.** See Week 2 homework. □

- Also, every subsequence of a Cauchy sequence is also a Cauchy sequence (why)? However, not every Cauchy sequence converges. An example is the sequence

  $$3, 3.1, 3.14, 3.141, 3.1415, \ldots$$

  in the metric space $(\mathbf{Q}, d)$ (the rationals $\mathbf{Q}$ with the usual metric $d(x, y) := |x - y|$). While this sequence is convergent in $\mathbf{R}$ (it converges to π), it does not converge in $\mathbf{Q}$ (since $\pi \notin \mathbf{Q}$, and a sequence cannot converge to two different limits). So in certain metric spaces, Cauchy sequences do not necessarily converge.

- However, if even part of a Cauchy sequence converges, then the entire Cauchy sequence must converge (to the same limit):

- **Lemma 3.** Let $(x^{(n)})_{n=m}^\infty$ be a Cauchy sequence in (X, d). Suppose that there is some subsequence $(x^{(n_j)})_{j=1}^\infty$ of this sequence which converges to a limit x_0 in X. Then the original sequence $(x^{(n)})_{n=m}^\infty$ also converges to x_0.

- **Proof.** See Week 2 homework. □

- As we have seen, some spaces, such as $(\mathbf{Q}, d)$, contain Cauchy sequences which do not converge. However, others do not:

- **Theorem 4.** Let $(\mathbf{R}, d)$ be the real line with the standard metric $d(x, y) := |x - y|$. Then every Cauchy sequence in $\mathbf{R}$ is convergent.

- **Proof.** See Theorem 30 of Week 3/4 notes to my 131AH class. Alternatively, read on. □

- Inspired by this, we make a definition.

- **Definition.** A metric space (X, d) is said to be *complete* iff every Cauchy sequence in (X, d) is in fact convergent in (X, d).

- Thus Theorem 4 states that the reals $(\mathbf{R}, d)$ are complete. The rationals $(\mathbf{Q}, d)$, on the other hand, are not complete.

- Complete metric spaces have some nice properties. For instance, they are *intrinsically closed*: no matter what space one places them in, they are always closed sets. More precisely:

- **Proposition 5.**

- (a) Let (X, d) be a metric space, and let $(Y, d|_{Y \times Y})$ be a subspace of (X, d). If $(Y, d|_{Y \times Y})$ is complete, then Y must be closed in X.

- (b) Conversely, suppose that (X, d) is a complete metric space, and Y is a closed subset of X. Then the subspace $(Y, d|_{Y \times Y})$ is also complete.

- **Proof.** See Week 2 homework. □

- In contrast, an incomplete metric space such as $(\mathbf{Q}, d)$ may be considered closed in some spaces (for instance, $\mathbf{Q}$ is closed in $\mathbf{Q}$) but not in others (for instance, $\mathbf{Q}$ is not closed in $\mathbf{R}$). Indeed, it turns out that given any incomplete metric space (X, d), there exists a *completion* $(\overline{X}, \overline{d})$, which is a larger metric space containing (X, d) which is complete, and such that X is not closed in $\overline{X}$ (indeed, the closure of X in $(\overline{X}, \overline{d})$ will be all of $\overline{X}$). For instance, the completion of $\mathbf{Q}$ is $\mathbf{R}$. We will not discuss completions further in this course, and refer the reader to Math 121, although we remark that the procedure for creating the completion $\overline{X}$ of an incomplete metric space X is actually a generalization of the procedure of creating the reals $\mathbf{R}$ from the rationals $\mathbf{Q}$ using formal limits of Cauchy sequences (as described in my 131AH notes).

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Compact metric spaces

- We now come to one of the most useful notions in point set topology, that of *compactness*. We begin by recalling a useful theorem from Math 131AH.

- **Definition.** A sequence $(x^{(n)})_{n=m}^{\infty}$ in a metric space (X, d) is said to be *bounded* iff there exists a ball $B(x, r)$ in (X, d) such that $x^{(n)} \in B(x, r)$ for all $n \geq m$. Similarly, a subset E of a metric space (X, d) is said to be *bounded* iff there exists a ball $B(x, r)$ in (X, d) such that $E \subseteq B(x, r)$. If a set or sequence is not bounded, it is said to be *unbounded*.

- Thus for instance, in the real line $\mathbf{R}$ with the standard metric d, the set $[1, 2)$ is bounded, but the set $[0, \infty)$ is not. (Can you prove these claims rigorously?).

- **Bolzano-Weierstrass theorem.** Let $(\mathbf{R}, d)$ be the real line with the standard metric. Then every bounded sequence in $(\mathbf{R}, d)$ has at least one convergent subsequence.

- **Proof.** See Week 6 notes of my Math 131AH class. $\square$

- This quickly extends to higher dimensions:

- **Corollary 6.** Let $(\mathbf{R}^n, d)$ be a Euclidean space with either the Euclidean metric $d = d_{l^2}$ or the taxicab metric d_{l^1}. Then every bounded sequence in $(\mathbf{R}^n, d)$ has at least one convergent subsequence.

- **Proof.** See Week 2 homework. $\square$

- This property of every sequence having a convergent subsequence is so important that we give it a name.

- **Definition.** A metric space (X, d) is said to be *compact* iff every sequence in (X, d) has at least one convergent subsequence.

- It is not easy to be compact: at a bare minimum, one must be both complete and bounded:

- **Proposition 7.** Let (X, d) be a compact metric space. Then (X, d) is both complete and bounded.

- **Proof.** See Week 2 homework. □

- It is also useful to talk about compact *sets*, rather than compact *metric spaces*.

- **Definition.** Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is *compact* iff the subspace $(Y, d|_{Y \times Y})$ of (X, d) is compact.

- From Proposition 7 and Proposition 5(a), we thus immediately obtain

- **Corollary 8.** Let (X, d) be a metric space, and let Y be a compact subset of X. Then Y is closed and bounded.

- The converse to this Corollary is true in $\mathbf{R}^n$:

- **Heine-Borel theorem.** Let $(\mathbf{R}^n, d)$ be a Euclidean space with either the Euclidean metric or the taxicab metric. Let E be a subset of $\mathbf{R}^n$. Then E is compact if and only if it is closed and bounded.

- **Proof.** See Week 2 homework. □

- However, the Heine-Borel theorem is not true for more general metrics. For instance, the integers $\mathbf{Z}$ with the discrete metric is closed (indeed, it is complete) and bounded, but not compact, since the sequence $1, 2, 3, 4, \ldots$ is in $\mathbf{Z}$ but has no convergent subsequence (why?). (One can generalize the Heine-Borel theorem if one replaces the concept of boundedness with a stronger one, that of *total boundedness*. We will not do so here, however, and refer the reader to Math 121 for more information).

- A key property of compact sets is the following, rather strange-sounding statement: every open cover of a compact set has a finite subcover.

- **Theorem 9.** Let (X,d) be a metric space, and let Y be a compact subset of X. Let $(V_\alpha)_{\alpha\in A}$ be a collection of open sets in X, and suppose that
$$Y \subseteq \bigcup_{\alpha\in A} V_\alpha.$$
(i.e. the collection $(V_\alpha)_{\alpha\in A}$ *covers* Y). Then there exists a *finite* subset F of A such that
$$Y \subseteq \bigcup_{\alpha\in F} V_\alpha.$$

- **Proof (Optional).** We assume for contradiction that there does not exist any finite subset F of A for which $Y \subset \bigcup_{\alpha\in F} V_\alpha$.

  Let y be any element of Y. Then y must lie in at least one of the sets V_α. Since each V_α is open, there must therefore be an $r > 0$ such that $B_{(X,d)}(y,r) \subseteq V_\alpha$. Now let $r(y)$ denote the quantity
$$r(y) := \sup\{r \in (0,\infty) : B_{(X,d)}(y,r) \subseteq V_\alpha \text{ for some } \alpha \in A\}.$$
  By the above discussion, we know that $r(y) > 0$ for all $y \in Y$. Now, let r_0 denote the quantity
$$r_0 := \inf\{r(y) : y \in Y\}.$$
  Since $r(y) > 0$ for all $y \in Y$, we have $r_0 \geq 0$. There are two cases: $r_0 = 0$ and $r_0 > 0$.

- **Case 1: $r_0 = 0$.** Then for every integer $n \geq 1$, there is at least one point y in Y such that $r(y) < 1/n$ (why?). We thus choose, for each $n \geq 1$, a point $y^{(n)}$ in Y such that $r(y^{(n)}) < 1/n$ (we can do this because of the axiom of choice). In particular we have $\lim_{n\to\infty} r(y^{(n)}) = 0$, by the squeeze test. The sequence $(y^{(n)})_{n=1}^\infty$ is a sequence in Y; since Y is compact, we can thus find a subsequence $(y^{(n_j)})_{j=1}^\infty$ which converges to a point $y_0 \in Y$.

- As before, we know that there exists some $\alpha \in A$ such that $y_0 \in V_\alpha$, and hence (since V_α is open) there exists some $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq V_\alpha$. Since $y^{(n)}$ converges to y_0, there must exist an $N \geq 1$ such that $y^{(n)} \in B(y_0, \varepsilon/2)$ for all $n \geq N$. In particular, by the triangle inequality we have $B(y^{(n)}, \varepsilon/2) \subseteq B(y_0, \varepsilon)$, and thus $B(y^{(n)}, \varepsilon/2) \subseteq V_\alpha$. By definition of $r(y^{(n)})$, this implies that $r(y^{(n)}) \geq \varepsilon/2$ for all $n \geq N$. But this contradicts the fact that $\lim_{n\to\infty} r(y^{(n)}) = 0$.

- **Case 2:** $r_0 > 0$. In this case we now have $r(y) > r_0/2$ for all $y \in Y$. This implies that for every $y \in Y$ there exists an $\alpha \in A$ such that $B(y, r_0/2) \in V_\alpha$ (why?).

  We now construct a sequence $y^{(1)}, y^{(2)}, \ldots$ by the following recursive procedure. We let $y^{(1)}$ be any point in Y. The ball $B(y^{(1)}, r_0/2)$ is contained in one of the V_α and thus cannot cover all of Y, since we would then obtain a finite cover, a contradiction. Thus there exists a point $y^{(2)}$ which does not lie in $B(y^{(1)}, r_0/2)$, so in particular $d(y^{(2)}, y^{(1)}) \geq r_0/2$. Choose such a point $y^{(2)}$. The set $B(y^{(1)}, r_0/2) \cup B(y^{(2)}, r_0/2)$ cannot cover all of Y, since we would then obtain two sets V_{α_1} and V_{α_2} which covered Y, a contradiction again. So we can choose a point $y^{(3)}$ which does not lie in $B(y^{(1)}, r_0/2) \cup B(y^{(2)}, r_0/2)$, so in particular $d(y^{(3)}, y^{(1)}) \geq r_0/2$ and $d(y^{(3)}, y^{(2)}) \geq r_0/2$. Continuing in this fashion we obtain a sequence $(y^{(n)})_{n=1}^\infty$ in Y with the property that $d(y^{(k)}, y^{(j)}) \geq r_0/2$ for all $k \geq j$. In particular the sequence $(y^{(n)})_{n=1}^\infty$ is not a Cauchy sequence, and in fact no subsequence of $(y^{(n)})_{n=1}^\infty$ can be a Cauchy sequence either. But this contradicts the assumption that Y is compact (by Lemma 2). $\square$

- It turns out that Theorem 9 has a converse: if Y has the property that every open cover has a finite sub-cover, then it is compact. (This is actually not all that hard to prove, but we will not do so here; a proof can be found in Math 121). In fact, this property is often considered the more fundamental notion of compactness than the sequence-based one. (For metric spaces, the two notions, that of compactness and sequential compactness, are equivalent, but for more general *topological spaces*, the two notions are slightly different. This however is beyond the scope of this course).

- Theorem 9 has an important corollary: that every nested sequence of non-empty compact sets is still non-empty.

- **Corollary 10.** Let (X, d) be a metric space, and let $K_1, K_2, K_3, \ldots$ be a sequence of non-empty compact subsets of X such that

$$K_1 \supset K_2 \supset K_3 \supset \ldots.$$

  Then the intersection $\bigcap_{n=1}^{\infty} K_n$ is non-empty.

- **Proof.** See Week 2 homework. □

- We close this section by listing some miscellaneous properties of compact sets.

- **Theorem 11.** Let (X, d) be a metric space.

- (a) If Y is a compact subset of X, and $Z \subseteq Y$, then Z is compact if and only if Z is closed.

- (b) If $Y_1, \ldots, Y_n$ are a finite collection of compact subsets of X, then their union $Y_1 \cup \ldots \cup Y_n$ is also compact.

- (c) Every finite subset of X is compact.

- **Proof.** See Week 2 homework. □

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Continuous functions

- You may recall the concept of a continuous function $f : \mathbf{R} \to \mathbf{R}$ from Math 131AH. We now generalize this concept to that of a continuous function $f : X \to Y$ from any metric space to any other metric space.

- **Definition.** Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is *continuous at* x_0 iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. We say that f is *continuous* iff it is continuous at every point $x \in X$.

- You should check this definition against the definition of continuity you learnt in Math 131AH and confirm that they are indeed consistent.

- Continuous functions are also sometimes called continuous maps. Mathematically, there is no distinction between the two terminologies.

- If $f : X \to Y$ is continuous, and K is any subset of X, then the restriction $f|_K : K \to Y$ of f to K is also continuous (why?).

- Continuous functions map convergent sequences to convergent sequences:

- **Theorem 12.** Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space. Let $f : X \to Y$ be a function, and let $x_0 \in X$ be a point in X. Then the following two statements are equivalent:

- (a) f is continuous at x_0.

- (b) Whenever $(x^{(n)})_{n=1}^\infty$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^\infty$ converges to $f(x_0)$ with respect to the metric d_Y.

- **Proof.** See Week 2 homework. □

- There is another classification of continuous functions, that the inverse image of an open set is always open:

- **Theorem 13.** Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space. Let $f : X \to Y$ be a function. Then the following four statements are equivalent:

- (a) f is continuous.

- (b) Whenever $(x^{(n)})_{n=1}^\infty$ is a sequence in X which converges to some point $x_0 \in X$ with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^\infty$ converges to $f(x_0)$ with respect to the metric d_Y.

- (c) Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X.

- (d) Whenever F is a closed set in Y, the set $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is a closed set in X.

- **Proof.** See Week 2 homework. □

- A quick corollary is that the composition of two continuous functions is continuous:

- **Corollary 14.** Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then the composition $g \circ f : X \to Z$, defined by $g \circ f(x) := g(f(x))$, is also continuous.

- **Proof.** See Week 2 homework. □

- It may seem strange that in Theorem 13(c), that continuity ensures that the *inverse* image of an open set is open. One may guess instead that the reverse should be true, that the *forward* image of an open set is open; but this is not true; see the homework.

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Continuity and compactness

- In this section (and in the rest of the notes), whenever we refer to a Euclidean space $\mathbf{R}^n$, we assume that $n \geq 1$ is an integer, and we give $\mathbf{R}^n$ the Euclidean metric d_{l^2}, unless otherwise specified. Similarly, we give the real line $\mathbf{R}$ the standard metric $d(x, y) := |x - y|$ unless otherwise specified.

- Continuous functions interact well with compact sets.

- **Theorem 15.** Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let $K \subseteq X$ be any compact subset of X. Then the image $f(K) := \{f(x) : x \in K\}$ of K is also compact.

- **Proof.** See Week 2 homework. □

- Combining this with the Heine-Borel theorem, we obtain

- **Corollary 16.** Let K be a closed and bounded subset of $\mathbf{R}^n$. Let $f : K \to \mathbf{R}^m$ be a continuous map from K to the Euclidean space $\mathbf{R}^m$. Then the image $f(K)$ is also closed and bounded. In particular, the function f is bounded on K.

- This corollary has an important consequence.

- **Definition.** Let $f : X \to \mathbf{R}$ be a function, and let $x_0 \in X$. We say that f *attains its maximum at* x_0 if we have $f(x_0) \geq f(x)$ for all $x \in X$ (i.e. f is larger (or equal to) at x_0 than at any other point in x). We say that f *attains its minimum at* x_0 if we have $f(x_0) \leq f(x)$ for all $x \in X$.

- **Maximum principle.** Let K be a closed and bounded subset of $\mathbf{R}^n$, and let $f : K \to \mathbf{R}$ be a continuous function. Then f attains its maximum at some point $x_{max} \in K$, and also attains its minimum at some point $x_{min} \in K$.

- **Proof.** See Week 2 homework. □

- You may have already encountered a one-dimensional special case of this maximum principle in 131AH.

- (The remainder of this section is optional.) Another advantage of continuous functions on compact sets is that they are *uniformly continuous*.

- **Definition.** Let $f : X \to Y$ be a map from one metric space (X, d_X) to another (Y, d_Y). We say that f is *uniformly continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ whenever $x, x' \in X$ are such that $d_X(x, x') < \delta$.

- Clearly every uniformly continuous function is continuous. The converse is not true (can you think of a counterexample?), unless the domain is compact:

- **Theorem 17.** If $f : X \to Y$ is a continuous map from one metric space (X, d_X) to another (Y, d_Y), and X is compact, then f is uniformly continuous.

- **Proof** Fix $\varepsilon > 0$. For every $x_0 \in X$, the function f is continuous at x_0. Thus there exists a $\delta(x_0) > 0$, depending on x_0, such that $d_Y(f(x), f(x_0)) < \varepsilon/2$ whenever $d_X(x, x_0) < \delta(x_0)$. In particular, by the triangle inequality this implies that $d_Y(f(x), f(x')) < \varepsilon$ whenever $x \in B_{(X,d_X)}(x_0, \delta(x_0)/2)$ and $d_X(x', x) < \delta(x_0)/2$. (Why?).

- Now consider the (possibly infinite) collection of balls $\{B_{(X,d_X)}(x_0, \delta(x_0)/2) : x_0 \in X\}$. Each ball is of course open, and the union of all these balls covers X, since each point x_0 in X is contained in its own ball $B_{(X,d_X)}(x_0, \delta(x_0)/2)$. Hence, by Theorem 9, there exists a finite number of points $x_1, \ldots, x_n$ such that the balls $B_{(X,d_X)}(x_j, \delta(x_j)/2)$ for $j = 1, \ldots, n$ cover X:

$$X \subseteq \bigcup_{j=1}^{n} B_{(X,d_X)}(x_j, \delta(x_j)/2).$$

  Now let $\delta := \min_{j=1}^{n} \delta(x_j)/2$. Since each of the $\delta(x_j)$ are positive, and there are only a finite number of j, we see that $\delta > 0$. Now let x, x' be any two points in X such that $d_X(x, x') < \delta$. Since the balls $B_{(X,d_X)}(x_j, \delta(x_j)/2)$ cover X, we see that there must exist $1 \leq j \leq n$ such that $x \in B_{(X,d_X)}(x_j, \delta(x_j)/2)$. Since $d_X(x, x') < \delta$, we have $d_X(x, x') < \delta(x_j)/2$, and so by the previous discussion we have $d_Y(f(x), f(x')) < \varepsilon$. We have thus found a δ such that $d_Y(f(x), f(x')) < \varepsilon$ whenever $d(x, x') < \delta$, and this proves uniform continuity as desired. □

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Continuity and connectedness

- We now describe another important concept in metric spaces, that of *connectedness*.

- **Definition** Let (X, d) be a metric space. We say that X is *disconnected* iff there exist disjoint non-empty open sets V and W in X such that $V \cup W = X$. (Equivalently, X is disconnected if and only if X contains a non-empty proper subset which is simultaneously closed and open). We say that X is *connected* iff is not disconnected.

- **Example.** Consider the set $X := [1, 2] \cup [3, 4]$, with the usual metric. This set is disconnected because the sets $[1, 2]$ and $[3, 4]$ are open relative to X (why?).

- Intuitively, a disconnected set is one which can be separated into two disjoint open sets; a connected set is one which cannot be separated in this manner.

- We defined what it means for a metric space to be connected; we can also define what it means for a set to be connected.

- **Definition.** Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is *connected* iff the metric space $(Y, d|_{Y \times Y})$ is connected, and we say that Y is *disconnected* iff the metric space $(Y, d|_{Y \times Y})$ is disconnected.

- On the real line, connected sets are easy to describe.

- **Theorem 18.** Let X be a subset of the real line $\mathbf{R}$. Then the following statements are equivalent.

- (a) X is connected.

- (b) Whenever $x, y \in X$ and $x < y$, the interval $[x, y]$ is also contained in X.

- **Proof.** First we show that (a) implies (b). Suppose that X is connected, and suppose for contradiction that we could find points $x < y$ in X such that $[x, y]$ is *not* contained in X. Then there exists a real number $x < z < y$ such that $z \notin X$. Thus the sets $(-\infty, z) \cap X$ and $(z, \infty) \cap X$ will cover X. But these sets are non-empty (because they contain x and y respectively) and are open relative to X, and so X is disconnected, a contradiction.

- Now we show that (b) implies (a). Let X be a set obeying the property (b). Suppose for contradiction that X is disconnected. Then there exist disjoint non-empty sets V, W which are open relative to X, such that $V \cup W = X$. Since V and W are non-empty, we may choose an $x \in V$ and $y \in W$. Since V and W are disjoint, we have $x \neq y$; without loss of generality we may assume $x < y$. By property (b), we know that the entire interval $[x, y]$ is contained in X.

- Now consider the set $[x, y] \cap V$. This set is both bounded and non-empty (because it contains x). Thus it has a supremum

$$z := \sup([x, y] \cap V).$$

Clearly $z \in [x, y]$, and hence $z \in X$. Thus either $z \in V$ or $z \in W$. Suppose first that $z \in V$. Then $z \neq y$ (since $y \in W$ and V is disjoint from W). But V is open relative to X, which contains $[x, y]$, so there is some ball $B_{([x,y],d)}(z, r)$ which is contained in V. But this contradicts the fact that z is the supremum of $[x, y] \cap V$. Now suppose that $z \in W$. Then $z \neq x$ (since $x \in V$ and V is disjoint from W). But W is open relative to X, which contains $[x, y]$, so there is some ball $B_{([x,y],d)}(z, r)$ which is contained in W. But this again contradicts the fact that z is the supremum of $[x, y] \cap V$. Thus in either case we obtain a contradiction, which means that X cannot be disconnected, and must therefore be connected. $\square$

- From Theorem 18, we see in particular that $\mathbf{R}$ is connected, as are any intervals (a, b), $[a, b]$, $(a, b]$, $[a, b)$, as well as point sets $\{a\}$, and half-infinite intervals $(a, +\infty)$, $[a, +\infty)$, $(-\infty, a)$, $(-\infty, a]$. Together with the empty set, these in fact form the only connected subsets of $\mathbf{R}$ (why?).

- Continuous functions map connected sets to connected sets:

- **Theorem 19** Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let E be any connected subset of X. Then $f(E)$ is also connected.

- **Proof.** See Week 2 homework. $\square$

- An important corollary of this result is the Intermediate value theorem.

- **Intermediate value theorem.** Let $f : X \to \mathbf{R}$ be a continuous map from one metric space (X, d_X) to the real line. Let E be any connected subset of X, and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, i.e. either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in E$ such that $f(c) = y$.

- **Proof.** See Week 2 homework. $\square$