Mathematics 131BH
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Instructions: Do nine out of the 12 problems; they are all of equal value. There is plenty of working space, and a blank page at the end. On the first page you will be supplied a list of standard definitions for easy reference.

Unless otherwise specified, we give spaces such as $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{Q}$ the usual metric $d(x, y) := |x - y|$. You are free to use the axiom of choice whenever you wish. (If you do not know what the axiom of choice is, please disregard this notice).

You may enter in a nickname if you want your final score posted. Good luck!

Name: ________________

Nickame: ________________

Student ID: ________________

Signature: ________________

Problem 1 (10 points). ________

Problem 2 (10 points). ________

Problem 3 (10 points). ________

Problem 4 (10 points). ________

Problem 5 (10 points). ________

Problem 6 (10 points). ________

Problem 7 (10 points). ________

Problem 8 (10 points). ________

Problem 9 (10 points). ________

Problem 10 (10 points). ________

Problem 11 (10 points). ________

Problem 12 (10 points). ________

Best 9 of 12 (90 points): ________

Definitions

- **Absolute integrability.** Let Ω be a measurable subset of $\mathbf{R}^n$. A measurable function $f : \Omega \to \mathbf{R}^*$ is said to be *absolutely integrable* if the integral $\int_\Omega |f|$ is finite.

- **Analyticity.** Let $(a-r, a+r)$ be an open interval, and let $f : E \to \mathbf{R}$ be a function defined on a set $E \subseteq \mathbf{R}$ which contains $(a-r, a+r)$. We say that f is *real analytic on* $(a-r, a+r)$ iff there exists a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ centered at a which has a radius of convergence greater than or equal to r, and which converges to f on $(a-r, a+r)$.

- **Boxes.** A (open) *box* B in $\mathbf{R}^n$ is any set of the form

$$B = \prod_{i=1}^{n} (a_i, b_i) := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in (a_i, b_i) \text{ for all } 1 \leq i \leq n\},$$

  where $b_i \geq a_i$ are real numbers. We define the *volume* $vol(B)$ of this box to be the number

$$vol(B) := \prod_{i=1}^{n} (b_i - a_i) = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n).$$

- **Compactness.** A metric space (X, d) is said to be *compact* iff every sequence in (X, d) has at least one convergent subsequence. If Y is a subset of X. We say that Y is *compact* iff the subspace $(Y, d|_{Y \times Y})$ of (X, d) is compact.

- **Connectedness.** Let (X, d) be a metric space. We say that X is *disconnected* iff there exist disjoint non-empty open sets V and W in X such that $V \cup W = X$. (Equivalently, X is disconnected if and only if X contains a non-empty proper set which is simultaneously closed and open). We say that X is *connected* iff is not disconnected.

- **Cover.** Let $\Omega \subseteq \mathbf{R}^n$ be a subset of $\mathbf{R}^n$. We say that a collection $(B_j)_{j \in J}$ of boxes *cover* Ω iff $\Omega \subseteq \bigcup_{j \in J} B_j$.

- **Differentiability.** Let E be a subset of $\mathbf{R}^n$, $f : E \to \mathbf{R}^m$ be a function, $x_0 \in E$ be a point, and let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. We say that f is *differentiable at x_0 with derivative L* if we have

$$\lim_{x \to x_0; x \in E - \{x_0\}} \frac{\|f(x) - (f(x_0) + L(x - x_0))\|}{\|x - x_0\|} = 0.$$

  Here $\|x\|$ is the length of x (as measured in the l^2 metric):

$$\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}.$$

- **Fourier transform.** For any function $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{R})$, and any integer $n \in \mathbf{Z}$, we define the n^{th} *Fourier coefficient of* f, denoted $\hat{f}(n)$, by the formula

$$\hat{f}(n) := \langle f, e_n \rangle = \int_{[0,1]} f(x)e^{-2\pi inx}\, dx.$$

  The function $\hat{f} : \mathbf{Z} \to \mathbf{C}$ is called the *Fourier transform* of f.

- **Integration of simple functions.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a simple function which is non-negative; thus f is measurable and the image $f(\Omega)$ is finite and contained in $[0, \infty)$. We then define the *Lesbegue integral* $\int_\Omega f$ of f on Ω by

$$\int_\Omega f := \sum_{\lambda \in f(\Omega); \lambda > 0} \lambda m(\{x \in \Omega : f(x) = \lambda\}).$$

- **Integration of non-negative measurable functions.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, \infty]$ be measurable and non-negative. Then we define the *Lebesgue integral* $\int_\Omega f$ of f on Ω to be

$$\int_\Omega f := \sup\{\int_\Omega s : s \text{ is simple and non-negative, and minorizes } f\}.$$

- **Integration of absolutely integrable functions.** Let $f : \Omega \to \mathbf{R}^*$ be an absolutely integrable function. We define the *Lebesgue integral* $\int_\Omega f$ of f to be the quantity

$$\int_\Omega f := \int_\Omega f^+ - \int_\Omega f^-.$$

- **Measurable functions.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. We say that f is *measurable* iff $f^{-1}(V)$ is measurable for every open set $V \subseteq \mathbf{R}^m$. A function $g : \Omega \to \mathbf{R}^*$ is said to be *measurable* iff we $f^{-1}((a, \infty))$ is measurable for every real number a.

- **Measurable sets.** Let E be a subset of $\mathbf{R}$. We say that E is *Lebesgue measurable*, or *measurable* for short, iff we have the identity

$$m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$$

  for every subset A of $\mathbf{R}$. If E is measurable, we define the *Lebesgue measure* of E to be $m(E) = m^*(E)$; if E is not measurable, we leave $m(E)$ undefined.

- **Simple functions.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a measurable function. We say that f is a *simple function* if the image $f(\Omega)$ is finite. In other words, there exists a finite number of real numbers $c_1, c_2, \ldots, c_N$ such that for every $x \in \Omega$, we have $f(x) = c_j$ for some $1 \leq j \leq N$.

- **Outer measure.** If Ω is a set, we define the *outer measure* $m^*(\Omega)$ of Ω to be the quantity

$$m^*(\Omega) := \inf\{\sum_{j=1}^{\infty} vol(B_j) : (B_j)_{j\in J} \text{ is a finite or countable cover of } \Omega \text{ by boxes}\}.$$

- **Uniform boundedness.** A sequence $(f_n)_{n=1}^{\infty}$ of functions from a metric space (X, d) to $\mathbf{R}$ is said to be *uniformly bounded* if there exists a constant $M > 0$ (which does not depend on n or x) such that $|f_n(x)| \leq M$ for all $x \in X$ and all positive integers n.

- **Uniform convergence** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges uniformly to* f *on* X if for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f^{(n)}(x), f(x)) < \varepsilon$ for every $n > N$ and $x \in X$.

Problem 1. Let (X, d) be a *connected* metric space, and let $f : X \to \mathbf{Z}$ be a *continuous* function from X to the integers. Show that f is constant. (Hint: Argue by contradiction, and consider the sets $\{x \in X : f(x) = n\}$ and $\{x \in X : f(x) \neq n\}$ for some integer n.)

Proof A. Suppose for contradiction that f is not constant; then f must take at least two different values on X; thus one can find $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$.

Let $n := f(x_1)$, and let $V := \{x \in X : f(x) = n\}$ and $W := \{x \in X : f(x) \neq n\}$. Then clearly $V \cap W = \emptyset$ and $V \cup W = X$, since for every $x \in X$, the number $f(x)$ is either equal to n or not equal to n, but not both. Also, $x_1 \in V$ and $x_2 \in W$, so V and W are both non-empty. Finally, since $V = f^{-1}(\{n\})$ and $W = f^{-1}(\mathbf{Z} - \{n\})$, then V is open (because $\{n\}$ is open in $\mathbf{Z}$, and f is continuous) and W is open (because $\mathbf{Z} - \{n\}$ is open in $\mathbf{Z}$, and f is continuous). This means that X is disconnected, contradiction. Thus f must be constant.

Proof B. Again suppose for contradiction that f is not constant; then the set $f(X)$ contains at least two distinct integers. Since X is connected and f is continuous, then $f(X)$ is also connected. Now let n be any element of $f(X)$. Then $\{n\}$ is clearly closed in $f(X)$; also, since $\{n\} = B(n, 1/2)$ (there are no other integers within $1/2$ of n), we see that $\{n\}$ is also open. Hence we have found a proper non-empty subset of $f(X)$ which is both open and closed, contradicting the fact that $f(X)$ is connected. Thus f must be constant.

Problem 2. Let (X, d) be a metric space, and for every positive integer n, let $f_n : X \to \mathbf{R}$ and $g_n : X \to \mathbf{R}$ be functions. Suppose that $(f_n)_{n=1}^{\infty}$ converges uniformly to another function $f : X \to \mathbf{R}$, and that $(g_n)_{n=1}^{\infty}$ converges uniformly to another function $g : X \to \mathbf{R}$. Suppose also that the functions $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are uniformly bounded (see Reference sheet). Prove that the functions $f_n g_n : X \to \mathbf{R}$ converge uniformly to $fg : X \to \mathbf{R}$.

Since $(f_n)_{n=1}^{\infty}$ are uniformly bounded, there exists an M such that $|f_n(x)| \leq M_1$ for all $x \in X$ and all $n \geq 1$. Similarly there exists an M_2 such that $|g_n(x)| \leq M_2$ for all $x \in X$.

Taking limits, we see that we also have $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all $x \in X$.

Now let $\varepsilon > 0$ be arbitrary. Since f_n converges uniformly to f, we know that there exists an integer $N_1 \geq 1$ such that $|f_n(x) - f(x)| \leq \varepsilon$ for all $n \geq N_1$. Similarly there exists an integer $N_2 \geq 1$ such that $|g_n(x) - g(x)| \geq \varepsilon$.

Now suppose that $n \geq \max(N_1, N_2)$. Then $|f_n(x) - f(x)| \leq \varepsilon$ and $|g(x)| \leq M_2$, thus $|f_n(x)g(x) - f(x)g(x)| \leq M_2\varepsilon$. Similarly, we have $|g_n(x) - g(x)| \leq \varepsilon$ and $|f_n(x)| \leq M_1$, hence $|f_n(x)g_n(x) - f_n(x)g(x)| \leq M_1\varepsilon$. By the triangle inequality we thus have

$$|f_n(x)g_n(x) - f(x)g(x)| \leq M_1\varepsilon + M_2\varepsilon$$

for all $x \in X$, or in other words

$$d_\infty(f_n g_n, fg) \leq (M_1 + M_2)\varepsilon \text{ for all } n \geq \max(N_1, N_2)$$

where d_∞ denotes the uniform metric. Thus we have $\limsup_{n\to\infty} d_\infty(f_n g_n, fg) \leq (M_1 + M_2)\varepsilon$. But ε was arbitrary, hence we have $\limsup_{n\to\infty} d_\infty(f_n g_n, fg) \leq 0$, and hence $\lim_{n\to\infty} d_\infty(f_n g_n, fg) = 0$ (since $d_\infty(f_n g_n, fg) \geq 0$). In other words, $f_n g_n$ converges uniformly to fg.

Problem 3. Let (X, d) be a metric space, let E be a non-empty compact subset of X, and let x_0 be a point in X. Show that there exists a point $x \in E$ such that

$$d(x_0, x) = \inf\{d(x_0, y) : y \in E\},$$

i.e. x is the closest point in E to x_0. (Hint: Write $R := \inf\{d(x_0, y) : y \in E\}$. Construct a sequence $(x^{(n)})_{n=1}^{\infty}$ in E such that $d(x_0, x^{(n)}) \leq R + \frac{1}{n}$, and then use the compactness of E).

Set $R := \inf\{d(x_0, y) : y \in E\}$. Note that R is finite because there is at least one point in E, and so the set $\{d(x_0, y) : y \in E\}$ contains at least one real number.

For any $n \geq 1$, observe that $R + \frac{1}{n}$ is larger than R, and hence cannot be a lower bound for $\{d(x_0, y) : y \in E\}$. Thus there exists a point $x^{(n)}$ in E such that $d(x_0, x^{(n)}) \leq R + \frac{1}{n}$. Also, by definition of R, we have $d(x_0, x^{(n)}) \geq R$. We now choose such a point $x^{(n)}$ for each integer n (this requires the axiom of choice). Since E is compact, there must exist a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of the sequence $(x^{(n)})_{n=1}^{\infty}$ which converges, in E, to another point $x \in E$.

Since $x^{(n_j)}$ converges to x, we have $d(x_0, x^{(n_j)})$ converging to $d(x_0, x)$ (see Q1 of Midterm 1). But since

$$R \leq d(x_0, x^{(n_j)}) \leq R + \frac{1}{n_j}$$

we see from the squeeze test that $d(x_0, x) = R$, and we are done.

Problem 4. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable function such that $f'(x) = 0$ for all $x \in \mathbf{R}^n$. Show that f is constant. (Hint: You may use the mean-value theorem or fundamental theorem of calculus for one-dimensional functions, but bear in mind that there is no direct analogue of these theorems for several-variable functions. I would not advise proceeding via first principles.)

Let $f_1 : \mathbf{R}^n \to \mathbf{R}$, $f_2 : \mathbf{R}^n \to \mathbf{R}$, $\ldots$, $f_m : \mathbf{R}^n \to \mathbf{R}$ be the m components of f. Since f is differentiable, all of its components are differentiable, and also $f_j'(x) = 0$ for all $1 \leq j \leq m$. (This comes directly from the definition of differentiability). To prove that f is constant, it suffices to show that each component f_j is constant.

Let $1 \leq j \leq m$, and let x, y be any two points in $\mathbf{R}^n$. We have to show that $f_j(x) = f_j(y)$. There are two ways to do this. One is to use directional derivatives. Let $g : \mathbf{R} \to \mathbf{R}$ be the function

$$g(t) := f_j(x + t(y - x)).$$

Then $g(0) = f_j(x)$ and $g(1) = f_j(y)$. Also, by the chain rule we see that

$$g'(t) = f_j'(x + t(y - x))(y - x) = 0(y - x) = 0$$

for all $t \in [0, 1]$. By the mean-value theorem we thus see that $g(1) = g(0)$, and we are done.

Another approach is to observe that since f_j is zero, all the partial derivatives $\frac{\partial f_j}{\partial x_k}$ are zero for each $k = 1, \ldots, n$. This means that as a function of x_k, f_j is constant. If we then write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we thus see that

$$f_j(x_1, x_2, \ldots, x_n) = f_j(y_1, x_2, \ldots, x_n)$$

since f is constant in the x_1 variable. Similarly we have

$$f_j(y_1, x_2, \ldots, x_n) = f(y_1, y_2, \ldots, x_n)$$

since f is constant in the x_2 variable. Continuing in this fashion one can eventually show that

$$f_j(x_1, \ldots, x_n) = f_j(y_1, \ldots, y_n)$$

as desired.

Note that one cannot directly apply the fundamental theorem of calculus because the functions were not assumed to be *continuously* differentiable.

Problem 5. Let $f, g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ be continuous, 1-periodic functions, and let $f * g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ be the periodic convolution

$$f * g(x) := \int_{[0,1]} f(y)g(x-y)\ dy.$$

Show that the Fourier coefficients of f, g, and $f * g$ are related by the formula

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n) \text{ for all } n \in \mathbf{Z}.$$

(Hint: Either first prove this for polynomials and then use the Weierstrass approximation theorem for trigonometric polynomials, or else use Fubini's theorem and an appropriate change of variables.)

Proof A. Fix n; we wish to show that

$$\widehat{f * g}(n) - \hat{f}(n)\hat{g}(n) = 0.$$

Let $\varepsilon > 0$. By the Weierstrass approximation theorem we can find a trigonometric polynomial P such that $\|g - P\|_\infty \leq \varepsilon$. From class notes we already know that

$$\widehat{f * P}(n) = \hat{f}(n)\hat{P}(n)$$

and thus

$$\widehat{f * g}(n) - \hat{f}(n)\hat{g}(n) = f * \widehat{(g - P)}(n) - \hat{f}(n)\hat{(g - P)}(n).$$

But

$$|\hat{(g - P)}(n)| = |\int_0^1 (g(x) - P(x))e^{-2\pi inx}\ dx| \leq \int_0^1 |g(x) - P(x)||e^{-2\pi inx}|\ dx$$

$$\leq \int_0^1 \varepsilon\ dx = \varepsilon.$$

Similarly

$$|f * \widehat{(g - P)}(n)| = |\int_0^1 (\int_0^1 f(y)(g(x-y) - P(x-y))\ dy)e^{-2\pi inx}\ dx|$$

$$\leq \int_0^1 (\int_0^1 |f(y)||g(x-y) - P(x-y)|\ dy)|e^{-2\pi inx}|\ dx$$

$$\leq \int_0^1 (\int_0^1 |f(y)|\varepsilon\ dy)\ dx = \varepsilon \int_0^1 |f(y)|\ dy.$$

Thus by the triangle inequality, we have

$$|\widehat{f * (g - P)}(n) - \hat{f}(n)\widehat{(g - P)}(n)| \leq \varepsilon \int_0^1 |f(y)| \, dy + \varepsilon |\hat{f}(n)|$$

and hence

$$|\widehat{f * g}(n) - \hat{f}(n)\hat{g}(n)| \leq \varepsilon (\int_0^1 |f(y)| \, dy + |\hat{f}(n)|).$$

But $\varepsilon > 0$ was arbitrary, and everything else does not depend on ε, so we have

$$|\widehat{f * g}(n) - \hat{f}(n)\hat{g}(n)| = 0$$

as desired.

Proof B. We can write

$$\widehat{f * g}(n) = \int_0^1 (\int_0^1 f(y)g(x-y) \, dy) e^{-2\pi i n x} \, dx$$

which is equal by Fubini's theorem to

$$\int_0^1 (\int_0^1 f(y)g(x-y)e^{-2\pi i n x} \, dx) dy$$

(note that the function $f(y)g(x-y)e^{-2\pi i n x}$ is continuous and bounded on $[0,1] \times [0,1]$, hence measurable and absolutely integrable). Making the change of variables $x = z + y$, this becomes

$$\int_0^1 (\int_{-y}^{1-y} f(y)g(z)e^{-2\pi i n(z+y)} \, dz) dy.$$

But since $g(z)e^{-2\pi i n(z+y)}$ is 1-periodic in z, the integral over $[-y, 1-y]$ is the same as the integral over $[0,1]$:

$$\int_0^1 (\int_0^1 f(y)g(z)e^{-2\pi i n(z+y)} \, dz) dy.$$

We now pull out all the factors from the inner integral that don't depend on z:

$$\int_0^1 (\int_0^1 g(z)e^{-2\pi i n z} \, dz) f(y) e^{-2\pi i n y} dy.$$

The inner integral is now $\hat{g}(n)$ and can be taken outside of the outer integral; then the outer integral is just $\hat{f}(n)$, and we are done.

Problem 6. Let $f : \mathbf{R} \to (0, \infty)$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbf{R}$. Show that $f(x) = Ce^x$ for some positive constant C; justify your reasoning. (Hint: there are basically three different proofs available. One proof uses the logarithm function, another proof uses the function e^{-x}, and a third proof uses power series. Of course, you only need to supply one proof.)

Proof A. Since f is analytic, it has a Taylor expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots.$$

By repeatedly differentiating the hypothesis $f' = f$ we see that $f'' = f' = f$, $f''' = f'' = f' = f$, etc. In particular, the quantities $f(0), f'(0), f''(0)$, etc. are all equal to the same number; call it C. Thus we have

$$f(x) = C + Cx + \frac{C}{2!}x^2 + \ldots;$$

comparing this with the definition of the exponential function we thus see that

$$f(x) = C\exp(x)$$

for all $x \in \mathbf{R}$. Since f is assumed to be positive, then C must also be positive.

Proof B. Since f is real analytic, it is differentiable. Since it is also positive, and the logarithm function is differentiable on $(0, \infty)$, we thus see from the chain rule that $\log f$ is also differentiable, and

$$(\log f)' = \frac{f'}{f} = 1.$$

Integrating this, we obtain $\log f = x + c$ for some real constant c; exponentiating this, we obtain $f = e^c e^x$. Setting $C = e^c > 0$ we obtain the result.

Proof C. Since f is real analytic, it is differentiable. Since e^{-x} is also differentiable, we see from the product rule that $e^{-x}f$ is differentiable, and

$$(e^{-x}f)' = -e^{-x}f + e^{-x}f' = 0$$

and hence $e^{-x}f$ is equal to some constant C; thus $f(x) = Ce^x$ for all x. Since f is positive, C must also be positive.

Problem 7. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuously differentiable function such that $f'(x)$ is an invertible linear transformation for every $x \in \mathbf{R}^n$. Show that whenever V is an open set in $\mathbf{R}^n$, that $f(V)$ is also open. (Hint: use the inverse function theorem).

Let V be any open set. We have to show that $f(V)$ is open; in other words, we need to show that for any point $y \in f(V)$ there exists a ball $B(y, r)$ which is also contained in $f(V)$.

Since $y \in f(V)$, we have $y = f(x_0)$ for some $x_0 \in V$. Since $f'(x_0)$ is invertible, we see from the inverse function theorem that there exists an open set $V' \subset V$ containing x_0, and an open set U containing y, such that f is a bijection from V' to U. In particular, $f(V') = U$, hence $f(V) \supset U$. But U is open and contains y, hence there is a ball $B(y, r)$ contained in U and hence in $f(V)$, as desired.

Problem 8. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be absolutely integrable, measurable functions such that $f(x) \leq g(x)$ for all $x \in \mathbf{R}$, and that $\int_{\mathbf{R}} f = \int_R g$. Show that $f(x) = g(x)$ for almost every $x \in \mathbf{R}$ (i.e. that $f(x) = g(x)$ for all $x \in \mathbf{R}$ except possibly for a set of measure zero).

Since f and g are absolutely integrable, so is $f - g$, and

$$\int_{\mathbf{R}} f - g = \int_R f - \int_R g = 0.$$

But $f - g$ is also non-negative, so by Proposition 3(a) of Week 10 notes, we have $f(x) - g(x) = 0$ for almost every x, and hence $f(x) = g(x)$ for almost every x.

Problem 9. Let A and B be subsets of $\mathbf{R}$ (not necessarily measurable) with finite outer measure, and let $A \times B := \{(a,b) : a \in A, b \in B\}$ be the Cartesian product of A and B (which is thus a subset of $\mathbf{R}^2$). Show that the two-dimensional outer measure $m_2^*(A \times B)$ of $A \times B$ is related to the one-dimensional outer measures $m_1^*(A)$, $m_1^*(B)$ of A and B by the inequality

$$m_2^*(A \times B) \leq m_1^*(A) m_1^*(B).$$

Let $\varepsilon > 0$. Then we may find a covering $(A_j)_{j \in J}$ of A by intervals A_j such that $\sum_j vol(A_j) \leq m_1^*(A) + \varepsilon$; this is from the definition of outer measure. Similarly we may find a covering $(B_k)_{k \in K}$ of K by intervals B_k such that $\sum_k vol(B_k) \leq m_1^*(B) + \varepsilon$.

Since the intervals $(A_j)_{j \in J}$ cover A, and the intervals $(B_k)_{k \in K}$ cover B, the boxes $(A_j \times B_k)_{j \in J; k \in K}$ cover $A \times B$ (since for every $(a,b) \in A \times B$, a belongs to one of the A_j and b belongs to one of the B_k, hence (a,b) belongs to one of the $A_j \times B_k$). Thus

$$m_2^*(A \times B) \leq \sum_{j \in J, k \in K} vol(A_j \times B_k).$$

But by definition of the volume of a box, we have $vol(A_j \times B_k) = vol(A_j) vol(B_k)$, and hence the right-hand side factorizes as

$$(\sum_{j \in J} vol(A_j))(\sum_{k \in K} vol(B_k))$$

(here we are using Fubini's theorem for infinite sums, see my 131AH notes). Thus we have

$$m_2^*(A \times B) \leq (m_1^*(A) + \varepsilon)(m_2^*(B) + \varepsilon).$$

Taking limits as $\varepsilon \to 0$ we obtain the result.

Remark: the same statement is true when A and B are allowed to have infinite outer measure, using the convention that $0 \times \infty = 0$, but the claim is a little trickier to prove there.

Remark: When A and B are measurable, then we in fact have equality, $m_2^*(A \times B) = m_1^*(A) m_1^*(B)$; this follows from Fubini's theorem. I do not know if one has equality in the non-measurable case; I suspect the answer is no.

Problem 10. For every positive integer n, let $f_n : \mathbf{R} \to [0, \infty)$ be a non-negative measurable function such that

$$\int_{\mathbf{R}} f_n \leq \frac{1}{4^n}.$$

Show that for every $\varepsilon > 0$, there exists a set E of Lebesgue measure $m(E) \leq \varepsilon$ such that $f_n(x)$ converges pointwise to zero for all $x \in \mathbf{R} \backslash E$. (Hint: first prove that $m(\{x \in \mathbf{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}) \leq \frac{\varepsilon}{2^n}$ for all $n = 1, 2, 3, \ldots$, and then consider the union of all the sets $\{x \in \mathbf{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}$.)

Let E_n be the set $E_n := \{x \in \mathbf{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}$; note that E_n is measurable. Then

$$\frac{1}{4^n} \geq \int_{\mathbf{R}} f_n \geq \int_{E_n} f_n \geq \int_{E_n} \frac{1}{\varepsilon 2^n} = m(E_n) \frac{1}{\varepsilon 2^n}$$

and hence $m(E_n) \leq \frac{\varepsilon}{2^n}$. By countable sub-additivity, if we then define $E := \bigcup_{n=1}^{\infty} E_n$, then we have

$$m(E) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Now let $x \notin E$. Then $x \notin E_n$ for any $n \geq 1$, and hence $f_n(x) \leq \frac{1}{\varepsilon 2^n}$ for all $n \geq 1$. In particular, since $f_n(x)$ is non-negative, we see from the squeeze test that $f_n(x)$ converges pointwise to zero as $n \to \infty$ for all $x \in \mathbf{R} \backslash E$. (In fact, the convergence is uniform. This is a special case of *Egoroff's theorem.*)

Problem 11. Let E be a measurable subset of $\mathbf{R}^n$, and let $-E$ be the set $-E := \{-x : x \in E\}$. Show that $-E$ is also measurable, and $m(-E) = m(E)$. (Hint: First show that $m^*(A) = m^*(-A)$ for any set $A \subseteq \mathbf{R}^n$).

First observe that for a box B, the negative box $-B$ is a translate of B (this is because $(-b, -a)$ is the same as (a, b) translated by $-a-b$) and hence $vol(-B) = vol(B)$. (Note that for more general sets A, it is not necessarily true that $-A$ is a translate of A).

Now let A be any subset of $\mathbf{R}^n$. Observe that a collection $(B_j)_{j \in J}$ of boxes cover A if and only if the collection $(-B_j)_{j \in J}$ of boxes cover $-A$. Note that the total volume of both collections are the same: $\sum_{j \in J} vol(B_j) = \sum_{j \in J} vol(-B_j)$. Taking infima over all collections, we thus see that $m^*(A) = m^*(-A)$ as desired.

Now suppose that E is measurable. We need to show that $-E$ is measurable, i.e. we need to show that

$$m^*(A) = m^*(A \cap -E) + m^*(A \backslash -E)$$

for every set A. But

$$m^*(A \cap -E) = m^*(-(A \cap -E)) = m^*(-A \cap E)$$

and

$$m^*(A \backslash -E) = m^*(-(A \backslash -E)) = m^*(-A \backslash E)$$

thus

$$m^*(A \cap -E) + m^*(A \backslash -E) = m^*(-A \cap E) + m^*(-A \backslash E) = m^*(-A) = m^*(A)$$

as desired, where we have used the fact that E is measurable.

Problem 12. Let $f : \mathbf{R} \to \mathbf{R}$ be an absolutely integrable function. Show that

$$\lim_{N\to\infty} \int_{[-N,N]} f = \int_{\mathbf{R}} f.$$

(Hint: use one of the convergence theorems from Week 10 notes).

We have

$$\int_{[-N,N]} f = \int_{\mathbf{R}} f\chi_{[-N,N]}.$$

The functions $f\chi_{[-N,N]}$ are all dominated by the absolutely integrable function $|f|$, and converge pointwise to f, so by the Lebesgue dominated convergence theorem we have $\lim_{N\to\infty} \int_{\mathbf{R}} f\chi_{[-N,N]} = \int_{\mathbf{R}} f$, as desired.
