Assignment 8 (Due May 30). Covers: Weeks 8/9 notes

Note: in this assignment you may use the axiom of choice whenever you wish. (If you do not know what the axiom of choice is, disregard this note).

In Q1-3. we assume that $\mathbf{R}^n$ is a Euclidean space, and we have a notion of measurable set in $\mathbf{R}^n$ (which may or may not co-incide with the notion of Lebesgue measurable set) and a notion of measure (which may or may not co-incide with Lebesgue measure) which obeys axioms (i)-(xiii).

- Q1. (a) Show that if $A_1 \subseteq A_2 \subseteq A_3 \ldots$ is an increasing sequence of measurable sets (so $A_j \subseteq A_{j+1}$ for every positive integer j), then we have $m(\bigcup_{j=1}^\infty A_j) = \lim_{j\to\infty} m(A_j)$.

- Q1. (b) Show that if $A_1 \supseteq A_2 \supseteq A_3 \ldots$ is an decreasing sequence of measurable sets (so $A_j \supseteq A_{j+1}$ for every positive integer j), and $m(A_1) < +\infty$, then we have $m(\bigcap_{j=1}^\infty A_j) = \lim_{j\to\infty} m(A_j)$.

- Q2. Show that for any positive integer $q > 1$, that the open box

  $$(0, 1/q)^n := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : 0 < x_j < 1/q \text{ for all } 1 \leq j \leq n\}$$

  and the closed box

  $$[0, 1/q]^n := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : 0 \leq x_j \leq 1/q \text{ for all } 1 \leq j \leq n\}$$

  both measure q^{-n}. (Hint: first show that $m((0, 1/q)^n) \leq q^{-n}$ for every $q \geq 1$ by covering $(0, 1)^n$ by some translates of $(0, 1/q)^n$. Using a similar argument, show that $m([0, 1/q]^n) \geq q^{-n}$. Then show that $m([0, 1/q]^n \backslash (0, 1/q)^n) \leq \varepsilon$ for every $\varepsilon > 0$, by covering the boundary of $[0, 1/q]^n$ with some very small boxes).

- Q3*. Show that for any box B, that $m(B) = vol(B)$. (Hint: first prove this when the co-ordinates a_j, b_j are rational, using Q2. Then take limits somehow (perhaps using Q1) to obtain the general case when the co-ordinates are real).

- Q4. Prove Lemma 1 of Week 8 notes. (Hint: You will have to use the definition of inf, and probably introduce a parameter ε. You may have to treat separately the cases when certain outer measures are equal to $+\infty$. (viii) can be deduced from (x) and (v). For (x), label the index set J as $J = \{j_1, j_2, j_3, \ldots\}$, and for each A_j, pick a covering of A_j by boxes whose total volume is no larger than $m^*(A_j) + \varepsilon/2^j$).

- Q5. Let $A \subseteq \mathbf{R}^2$ be the set $A := [0,1]^2 \backslash \mathbf{Q}^2$; i.e A consists of all the points (x, y) in $[0,1]^2$ such that x and y are not both rational. Show that A is measurable and $m(A) = 1$, but that A has no interior points. (Hint: it's easier to use the properties of outer measure and measure derived in the notes and homework, than to try to do this problem from first principles).

- Q6. Let A be a subset of $\mathbf{R}^n$, and let B be a subset of $\mathbf{R}^m$. Note that the Cartesian product $\{(a, b) : a \in A, b \in B\}$ is then a subset of $\mathbf{R}^{n+m}$. Show that $m^*_{n+m}(A \times B) \leq m^*_n(A) m^*_m(B)$. (It is in fact true that $m^*_{n+m}(A \times B) = m^*_n(A) m^*_m(B)$, but this is substantially harder to prove).

- Q7 (a). If A is an open interval in $\mathbf{R}$, show that $m^*(A) = m^*(A \cap (0, \infty)) + m^*(A \backslash (0, \infty))$.

- Q7 (b). If A is an open box in $\mathbf{R}^n$, and E is the half-plane $E := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}$, show that $m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$. (Hint: use Q7(a)).

- Q7 (c). Using part (b), prove Lemma 6 of Week 8 notes.

- Q8. Prove Lemma 7 of Week 8 notes. (Hints: For (c), first prove that

  $$m^*(A) = m^*(A \cap E_1 \cap E_2) + m^*(A \cap E_1 \backslash E_2) + m^*(A \cap E_2 \backslash E_1) + m^*(A \backslash (E_1 \cup E_2)).$$

  A Venn diagram may be helpful. Also you may need the finite sub-additivity property. Use (c) to prove (d), and use (bd) and the various versions of Lemma 6 to prove (e)).

- Q9 (a). Prove Lemma 8 of Week 8 notes. (Hint: first prove this for unions of two disjoint sets).

- Q9 (b). Using Lemma 8, prove Corollary 9 of Week 8 notes.

- Q10. Prove Lemma 11 of Week 8 notes. (Hint: for the countable union problem, write $J = \{j_1, j_2, \ldots\}$, write $F_N := \bigcup_{k=1}^N \Omega_{j_k}$, and write $E_N := F_N \backslash F_{N-1}$, with the understanding that F_0 is the empty set. Then apply Lemma 10. For the countable intersection problem, use what you just did and Lemma 7(a)).