Assignment 5 (Due May 9). Covers: Weeks 4-5 notes

- Q1. Prove Theorem 12 from Weeks 4-5 notes. (Hint: for (ab), use the root test (see e.g. Week 5 of my 131AH notes, or Theorem 3.33 of Rudin). For (c), use the Weierstrass M-test. For (d), use Theorem 1 from Weeks 4-5 notes. For (e), use Corollary 10 from Weeks 4-5 notes).
- Q2. Give examples of a formal power series $\sum_{n=0}^{\infty} c_n x^n$ centered at 0 with radius of convergence 1, which
- (a) diverges at both $x = 1$ and $x = -1$;
- (b) diverges at $x = 1$ but converges at $x = -1$;
- (c) converges at $x = 1$ but diverges at $x = -1$;
- (d) converges at both $x = 1$ and $x = -1$.
- (e) converges pointwise on $(-1, 1)$, but does not converge uniformly on $(-1, 1)$.
- Q3 (a). Prove Proposition 13 from Weeks 4-5 notes. (Hint: Use Theorem 12(d), and induct on k).
- Q3 (b). Using Proposition 13, prove Corollary 14 from Weeks 4-5 notes.
- Q4. Do Question 1 of Chapter 8 from page 196 from the textbook; for this question you may use any property of the exponential function that you know from other courses. Conclude using Taylor's formula that this function f is not analytic on any interval $(-r, r)$ centered at 0.
- Q5. Prove the summation by parts formula. (Hint: first work out the relationship between the partial sums $\sum_{n=0}^{N}(a_{n+1} - a_n)b_n$ and $\sum_{n=0}^{N} a_{n+1}(b_{n+1} - b_n)$).
- Q6. Prove Theorem 17 from Weeks 4-5 notes. (Hints: For part (a), use the ratio test. For parts (bc), use Theorem 12. For part (d), use Theorem 16. For part (e), use part (d). For part (f), use part (d), and prove that $\exp(x) > 1$ when x is positive. You may find the binomial formula $(x + y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}$ to be useful.)

- Q7. (a) Show that for every integer $n \geq 3$, we have

$$0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots < \frac{1}{n!}.$$

(Hint: first show that $(n+k)! > 2^k n!$ for all $k = 1, 2, 3, \ldots$. Conclude that $n!e$ is not an integer for every $n \geq 3$. Deduce from this that e is irrational. (Hint: prove by contradiction).

- Q7 (b)*. Modifying the argument in (a), show that e^m is also irrational for any positive integer m. Also show that $\sin(m)$ is irrational for any positive integer m.

- Q7 (c). Using Q7(b), conclude that π is irrational. (Hint: if $\pi = p/q$ for some integers p, q, what does this say about $\sin(q\pi)$?)

- Q8. Prove Proposition 18 from Weeks 4-5 notes. (Hint: first prove the claim when x is a natural number. Then prove it when x is an integer. Then prove it when x is a rational number. Then use the fact that real numbers are the limits of rational numbers to prove it for all real numbers. You may find the exponent laws (see e.g. Week 5 notes from Math 131AH) to be useful).

- Q9. Prove Theorem 19 from Weeks 4-5 notes. (Hints: for part (a), use the inverse function theorem or the chain rule. For parts (bcd), use Theorem 17 and the exponent laws. For part (e), start with the geometric series formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and integrate using Theorem 12).

- Q10 (a). Prove Theorem 20 from Weeks 4-5 notes. (Hint: Write everything in terms of exponentials whenever possible).

- Q10 (b). Prove Theorem 22 from Weeks 4-5 notes. (Hint: For (c), you may wish to first compute $\sin(\pi/2)$ and $\cos(\pi/2)$, and then link $\cos(x)$ to $\sin(x + \pi/2)$).