Assignment 4 (Due May 2). Covers: Weeks 4-5 notes

- Q1. Complete the proof of Theorem 1 from Weeks 4-5 notes.
- Q2. Prove Corollary 2 from Weeks 4-5 notes.
- Q3*. For this question you may use any fact about trigonometric functions from your lower-division classes. Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) := \sum_{n=1}^{\infty} 4^{-n} \cos(32^n \pi x)$. As discussed in the Weeks 4-5 notes, this function is continuous on $\mathbf{R}$.
- (a) Show that for every integer j and every integer $m \geq 1$, we have

$$|f(\frac{j+1}{32^m}) - f(\frac{j}{32^m})| \geq 4^{-m}.$$

(Hint: use the identity

$$\sum_{n=1}^{\infty} a_n = (\sum_{n=1}^{m-1} a_n) + a_m + \sum_{n=m+1}^{\infty} a_n$$

for certain sequences a_n. Also, use the fact that the cosine function is periodic with period 2π, as well as the geometric series formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for any $|r| < 1$. Finally, you will need the inequality $|\cos(x) - \cos(y)| \leq |x - y|$ for any real numbers x and y; this can be proven by using the mean value theorem, or the fundamental theorem of calculus.)

- (b) Using (a), show that for every real number x_0, the function f is not differentiable at x_0. (Hint: for every x_0 and every $m \geq 1$, there exists an integer j such that $j \leq 32^m x_0 \leq j + 1$).
- (c) Explain briefly why the result in (b) does not contradict Corollary 2 from Weeks 4-5 notes.
- Q4. Prove Lemma 3 from Weeks 4-5 notes.
- Q5(a) Prove that for any real number $0 \leq y \leq 1$ and any natural number $n \geq 0$, that $(1-y)^n \geq 1-ny$. (Hint: induct on n. Alternatively, differentiate with respect to y.).

- Q5(b) Show that $\int_{-1}^{1}(1-x^2)^n \, dx \geq \frac{1}{\sqrt{n}}$. (Hint: for $|x| \leq 1/\sqrt{n}$, use part (a); for $|x| \geq 1/\sqrt{n}$, just use the fact that $(1-x^2)$ is positive. It is also possible to proceed via trigonometric substitution, but I would not recommend this unless you know what you are doing).

- Q5(c) Prove Lemma 4 from Weeks 4-5 notes. (Hint: Choose $f(x)$ to equal $c(1-x^2)^N$ for $x \in [-1,1]$ and to equal zero for $x \notin [-1,1]$, where N is a large number N, where c is chosen so that f has integral 1, and use (b)).

- Q6. Let $f : \mathbf{R} \to \mathbf{R}$ be a compactly supported, continuous function. Show that f is bounded and uniformly continuous. (Hint: Use Corollary 16 and Theorem 17 from Week 2).

- Q7*. Prove Proposition 5 from Weeks 4-5 notes. (Hint: To show that $f * g$ is continuous, use Q6).

- Q8(a). Let g be an (ε, δ) approximation to the identity. Show that $1 - 2\varepsilon \leq \int_{[-\delta,\delta]} g \leq 1$.

- Q8*(b). Prove Lemma 7 from Weeks 4-5 notes. (Hint: begin with the identity

$$f * g(x) = \int f(x-y)g(y) \, dy = \int_{[-\delta,\delta]} f(x-y)g(y) \, dy$$

$$+ \int_{[\delta,1]} f(x-y)g(y) \, dy + \int_{[-1,-\delta]} f(x-y)g(y) \, dy.$$

The idea is to show that the first integral is close to $f(x)$, and that the second and third integrals are very small. To achieve the former task, use Q8(a) and the fact that $f(x)$ and $f(x-y)$ are within ε of each other; to achieve the latter task, use property (c) of the approximation to the identity and the fact that f is bounded.)

- Q9. Prove Corollary 8 from Weeks 4-5 notes. (Hint: Combine Q6, Lemma 4, Lemma 6, and Lemma 7).

- Q10. Prove Lemma 9 from Weeks 4-5 notes.