Assignment 3 (Due April 25). Covers: Week 3 notes

Note: Some of the theorems to be proved here are also proved in the textbook; it is acceptable to use the proofs in the textbook provided that you rephrase them in your own words. Unless otherwise specified, $\mathbf{R}$ (and various subsets of $\mathbf{R}$) are always assumed to be given the standard metric $d(x, y) := |x - y|$.

- Q1. Prove Proposition 1 from Week 3 notes. (Hint. The proof of this Proposition is very similar to that of Theorem 12 in Week 2 notes).

- Q2 (a). Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function from X to Y. Show that if $f^{(n)}$ converges uniformly to f, then $f^{(n)}$ also converges pointwise to f.

- Q2 (b). For each integer $n \geq 1$, let $f^{(n)} : (-1, 1) \to \mathbf{R}$ be the function $f^{(n)}(x) := x^n$. Prove that $f^{(n)}$ does not converge uniformly to any function $f : (-1, 1) \to \mathbf{R}$ (this is despite $f^{(n)}$ converging pointwise, as mentioned in the notes). Justify your reasoning.

- Q2 (c). Let $g : (-1, 1) \to \mathbf{R}$ be the function $g(x) := x/(1 - x)$. With the notation as in Q2(b), show that $\sum_{n=1}^{\infty} f^{(n)}$ converges pointwise to g, but does not converge uniformly to g, on the open interval $(-1, 1)$. What would happen if we replaced the open interval $(-1, 1)$ with the closed interval $[-1, 1]$?

- Q3. Prove Theorem 2 from Week 3 notes. Explain briefly why your proof requires uniform convergence, and why pointwise convergence would not suffice. (Hint: it is easiest to use the "epsilon-delta" definition of continuity. You may find the triangle inequality

  $$d_Y(f(x), f(x_0)) \leq d_Y(f(x), f^{(n)}(x)) + d_Y(f^{(n)}(x), f^{(n)}(x_0)) + d_Y(f^{(n)}(x_0), f(x_0))$$

  useful. Also, you may need to divide ε as $\varepsilon = \varepsilon/3 + \varepsilon/3 + \varepsilon/3$. Finally, it is possible to prove Theorem 2 from Proposition 4, but you may find it easier conceptually to prove Theorem 2 first.)

- Q4. Prove Proposition 4 from Week 3 notes. (This is very similar to Theorem 2. Theorem 2 cannot be used to prove Proposition 4, however it is possible to use Proposition 4 to prove Theorem 2).

- Q5 (a). Prove Proposition 5 from Week 3 notes. (Again, this is similar to Theorem 2 and Proposition 4, although the statements are slightly different, and so one cannot deduce this directly from the other two results).

- Q5 (b). Give an example to show that Proposition 5 fails if the phrase "converges uniformly" is replaced by "converges pointwise". (Hint: some of the examples already given in the notes will already work here).

- Q6 (a). Prove Proposition 6 from Week 3 notes.

- Q6 (b). Give an example to show that Proposition 6 fails if the phrase "converges uniformly" is replaced by "converges pointwise". (Hint: Note that the ball $B_{(Y,d_Y)}(y_0, R)$ is allowed to be different for different values of n).

- Q7 (a). Let (X, d_X) and (Y, d_Y) be metric spaces. Show that the space $B(X; Y)$ defined in the notes, with the metric $d_{B(X;Y)}$, is indeed a metric space.

- Q7(b). Prove Proposition 7 from Week 3 notes.

- Q8. Prove Theorem 8 from Week 3 notes. (This is somewhat similar to the proof of Theorem 2, though not identical).

- Q9 (a). Let $f^{(1)}, \ldots, f^{(N)}$ be a finite sequence of continuous functions from a metric space (X, d_X) to $\mathbf{R}$. Show that $\sum_{i=1}^{N} f^{(i)}$ is also continuous.

- Q9 (b). Prove the Weierstrass M-test. (Hint: First show that the sequence $\sum_{i=1}^{N} f^{(i)}$ is a Cauchy sequence in $C(X; \mathbf{R})$; you may need to review some material from 131AH on absolute convergence to do so. Then use Theorem 8 from Week 3 notes and Theorem 4 from Week 2 notes (which asserts that $\mathbf{R}$ is complete)).

- Q10. Prove Corollary 10 from Week 3 notes.