Assignment 1 (Due April 11). Covers: Week 1 notes

Note: For these assignments you may freely use any material from the textbook (or any other book) or from other courses, especially from 131AH.

- Q1. Prove Lemma 1 from Week 1 notes.
- Q2 (a). Let $n \geq 1$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Verify the identity

$$(\sum_{i=1}^n a_i b_i)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = (\sum_{i=1}^n a_i^2)(\sum_{j=1}^n b_j^2),$$

and conclude the *Cauchy-Schwarz inequality*

$$|\sum_{i=1}^n a_i b_i| \leq (\sum_{i=1}^n a_i^2)^{1/2} (\sum_{j=1}^n b_j^2)^{1/2}.$$

Then use the Cauchy-Schwarz inequality to prove the *triangle inequality*

$$(\sum_{i=1}^n (a_i + b_i)^2)^{1/2} \leq (\sum_{i=1}^n a_i^2)^{1/2} + (\sum_{j=1}^n b_j^2)^{1/2}.$$

(Hint: square both sides).

- Q2(b). Let d_{l^2} be the Euclidean metric on $\mathbf{R}^n$. Use Q2(a) to show that $(\mathbf{R}^n, d_{l^2})$ is a metric space.
- Q3. Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function.
- Q3(a) Give an example of a pair (X, d) which obeys axioms (i), (ii), (iii) but not (iv). (Hint: try examples where X is a finite set).
- Q3(b) Give an example of a pair (X, d) which obeys axioms (i), (iii), (iv), but not (ii).
- Q3(c) Give an example of a pair (X, d) which obeys axioms (i), (ii), (iv), but not (iii).
- Q3(d) Give an example of a pair (X, d) which obeys axioms (ii), (iii), (iv), but not (i). (Hint: modify the discrete metric).

- Q4. Prove Proposition 2 from Week 1 notes.
- Q5. Prove Proposition 3 from Week 1 notes.
- Q6. Prove Proposition 4 from Week 1 notes.
- Q7. Prove Proposition 5 from Week 1 notes. (Note: to obtain (c) from (a) or (b) you will need to use the axiom of choice. If you do not know what the axiom of choice is, please disregard this note).
- Q8. Prove Proposition 6 from Week 1 notes.
- Q9. Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$, and let C be the closed ball $C := \{x \in X : d(x, x_0) \leq r\}$.
- (a) Show that $\overline{B} \subseteq C$.
- (b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B}$ is *not* equal to C.
- Q10. Prove Proposition 7(b) from Week 1 notes.