Math 131AH - Weeks 3 and 4

Textbook pages: 24-30, 47-58. (Optional additional reading: 30-32, esp. the part relating to limit points).

Topics covered:

- Countable and uncountable sets
- The uncountability of the reals
- Sequences of reals
- Limits and limit points

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More on cardinality

- In the last week's notes, we introduced some basic notions of cardinality of sets. Two sets were said to have equal cardinality if there existed a bijection from one set to the other. A set was said to be cardinality n for some natural number n, if it had equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq n\}$. A set is finite if it has cardinality n for some natural number n; otherwise, it is infinite.
- We already have a few results on finite sets; for instance, we know that a finite set has exactly one cardinality. Now we state a few more basic results.
- **Proposition 1.** Let n, m be natural numbers.
- (a) Let X be a set of cardinality n, and let $x \notin X$ be another object which is not an element of X. Then $X \cup \{x\}$ has cardinality $n+1$.
- (b) Let X be a set of cardinality n, and Y be a set of cardinality m. Then $X \cup Y$ is finite and has cardinality at most $n+m$. If in addition X and Y are disjoint (i.e. $X \cap Y = \emptyset$), then $X \cup Y$ has cardinality exactly $n+m$.
- (c) Let X be a set of cardinality n, and let Y be a subset of X. Then Y is finite, and has cardinality m for some $0 \leq m \leq n$. If in addition $Y \neq X$ (i.e. Y is a proper subset of X), then we have $m < n$.

- (d) If X is a finite set of cardinality n, and $f : X \to Y$ is a function, then $f(X)$ is a finite set of cardinality less than or equal to n. If in addition f is one-to-one, then $f(X)$ has cardinality exactly n.

- (A remark on notation: it is a convention that functions and objects are generally represented using lower case letters such as f and x, while sets and spaces are generally represented using upper case letters such as X and Y. However, this convention is not universally followed).

- **Proof.** See Week 3 homework. $\square$

- As one consequence of Proposition 1, every subset of a finite set is finite, the union of two finite sets is finite, and the image of any finite set under any function is still finite.

- From Proposition 1(c) we know that if X is a finite set, and Y is a proper subset of X, then Y does not have equal cardinality with X. However, this is not the case for infinite sets. For instance, from last week's notes we know that the set $\mathbf{N}$ of natural numbers is infinite. The set $\mathbf{N} - \{0\}$ is also infinite (why?), and is a proper subset of $\mathbf{N}$. However, the set $\mathbf{N} - \{0\}$, despite being "smaller" than $\mathbf{N}$, still has the same cardinality as $\mathbf{N}$, because the function $f : \mathbf{N} \to \mathbf{N} - \{0\}$ defined by $f(n) := n + 1$, is a bijection from $\mathbf{N}$ to $\mathbf{N} - \{0\}$. (Why?). This is one characteristic of infinite sets; they are often of the same cardinality as some of their subsets.

- **Definition.** A set X is said to be *countably infinite* (or just *countable*) iff it has equal cardinality with the natural numbers $\mathbf{N}$. A set X is said to be *at most countable* iff it is either countable or finite.

- Thus, $\mathbf{N}$ is countable, and so is $\mathbf{N} - \{0\}$. Another example of a countable set is the even natural numbers $\{2n : n \in \mathbf{N}\}$, since the function $f(n) := 2n$ provides a bijection between $\mathbf{N}$ and the even natural numbers (why?).

- Let X be a countable set. Then, by definition, we know that there exists a bijection $f : \mathbf{N} \to X$ from $\mathbf{N}$ to X. Thus, every element of X can be written in the form $f(n)$ for exactly one natural number n.

Informally, we thus have

$$X = \{f(0), f(1), f(2), f(3), \ldots\}.$$

Thus, a countable set can be arranged in a sequence, so that we have a zeroth element $f(0)$, followed by a first element $f(1)$, then a second element $f(2)$, and so forth, in such a way that all these elements $f(0), f(1), f(2), \ldots$ are all distinct, and together they fill out all of X. (This is why these sets are called *countable*; because we can literally count them one by one, starting from $f(0)$, then $f(1)$, etc.).

- Viewed in this way, it is clear why the natural numbers

$$\mathbf{N} = \{0, 1, 2, 3, \ldots\},$$

the positive integers

$$\mathbf{N} - \{0\} = \{1, 2, 3, \ldots\},$$

and the even natural numbers

$$\{0, 2, 4, 6, 8, \ldots\}$$

are countable. However, it is not as obvious whether the integers

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

or the rationals

$$\mathbf{Q} = \{0, 1/4, -2/3, \ldots\}$$

or the reals

$$\mathbf{R} = \{0, \sqrt{2}, -\pi, 2.5, \ldots\}$$

are countable or not (can we arrange them in a sequence $f(0), f(1), f(2), \ldots$?). We will answer these questions shortly.

- From Proposition 29 and Theorem 32 from last week's notes, we know that countable sets are infinite; however it is not so clear whether all infinite sets are countable. Again, we will answer those questions shortly.

- First, we give some properties of countable sets.

- **Proposition 2.** Let X be a subset of the natural numbers $\mathbf{N}$. Then X is at most countable. In particular, every infinite subset of the natural numbers is countable.

- To prove this, we first need the following principle.

- **Well-ordering principle.** Let X be a non-empty subset of the natural numbers $\mathbf{N}$. Then there exists exactly one element $n \in X$ such that $n \leq m$ for all $m \in X$. (In other words, every non-empty set of natural numbers has a minimum element).

- **Proof of Well-ordering principle.** See Week 3 homework. $\square$

- We will refer to the element n given by the well-ordering principle as the *minimum* of X, and write it as $\min(X)$. Thus for instance the minimum of the set $\{2, 4, 6, 8, \ldots\}$ is 2.

- **Proof of Proposition 2.** We will give an incomplete sketch of the proof, with some gaps marked by a question mark (?); these gaps will be filled in by the homework.

- Let X be a subset of $\mathbf{N}$. We need to show that X is at most countable. If X is finite, then we are done; so let us assume that X is infinite. Our task is then to show that X is countable, i.e. that X has the same cardinality as $\mathbf{N}$.

- We now define a sequence $a_0, a_1, a_2, \ldots$ of natural numbers, and a sequence $A_0, A_1, A_2, \ldots$ of *sets* of natural numbers, as follows.

- First, we initialize A_0 to be equal to X, $A_0 := X$, and then set a_0 to equal the minimum of A_0, $a_0 := \min(A_0)$. (Thus a_0 is the smallest element of X). Now suppose recursively that A_n and a_n have already been defined for some natural number n; then we set $A_{n++} := A_n - \{a_n\}$, and then set $a_{n++} := \min(A_{n++})$.

- (Intuitively speaking, a_0 is the smallest element of X; a_1 is the second smallest element of X, i.e. the smallest element of X once a_0 is removed; a_2 is the third smallest element of X; and so forth).

- There is a possible problem with this definition: the well-ordering principle only works for non-empty sets, and we have not checked that the sets A_n are non-empty. However, it is possible to show(?) that the sets A_n are always well-defined and always non-empty; indeed, one can show(?) that the A_n are always infinite.

- One can show(?) that a_n is an increasing sequence, i.e.

$$a_0 < a_1 < a_2 < \ldots$$

  and in particular that(?) $a_n \neq a_m$ for all $n \neq m$. Also, one can show(?) that each A_n is a subset of X, and hence that $a_n \in X$ for each natural number n.

- Now define the function $f : \mathbf{N} \to X$ by $f(n) := a_n$. From the previous paragraph we know that f is one-to-one. Now we show that f is onto. In other words, we claim that for every $x \in X$, there exists an n such that $a_n = x$.

- Let $x \in X$. Suppose for contradiction that $a_n \neq x$ for every natural number n. Then this implies(?) that $x \in A_n$ for every natural number n. Since $a_n = \min(A_n)$, this implies that $x \geq a_n$ for every natural number n. However, since a_n is an increasing sequence, we have $a_n \geq n$ (?), and hence $x \geq n$ for every natural number n. In particular we have $x \geq x{++}$, which is a contradiction. Thus we must have $a_n = x$ for some natural number n, and hence f is onto.

- Since $f : \mathbf{N} \to X$ is both one-to-one and onto, it is a bijection. Thus X is countable as desired. $\square$

- **Corollary 3.** If X is a countable set, and Y is a subset of X, then Y is at most countable.

- **Proof.** Since X is a countable set, there is a bijection $f : X \to \mathbf{N}$ between X and $\mathbf{N}$. Since Y is a subset of X, and f is a bijection from X and $\mathbf{N}$, then when we restrict f to Y, we obtain a bijection between Y and $f(Y)$ (why is this a bijection?). Thus $f(Y)$ has equal cardinality with Y. But $f(Y)$ is a subset of $\mathbf{N}$, and hence at most countable by Proposition 2. Hence Y is also at most countable. $\square$

- **Proposition 4.** Let Y be a set, and let $f : \mathbf{N} \to Y$ be a function. Then $f(\mathbf{N})$ is at most countable.

- **Proof.** See Week 3 homework. $\square$

- **Corollary 5.** Let X be a countable set, and let $f : X \to Y$ be a function. Then $f(X)$ is at most countable.

- **Proof.** See Week 3 homework. $\square$

- **Proposition 6.** Let X be a countable set, and let Y be a countable set. Then $X \cup Y$ is a countable set.

- **Proof.** See Week 3 homework. $\square$

- **Corollary 7.** The integers $\mathbf{Z}$ are countable.

- **Proof.** We already know that the natural numbers $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ are countable. The set $-\mathbf{N}$ defined by

$$-\mathbf{N} := \{-n : n \in \mathbf{N}\} = \{0, -1, -2, -3, \ldots\}$$

is also countable, since the map $f(n) := -n$ is a bijection between $\mathbf{N}$ and this set. Since the integers are the union of $\mathbf{N}$ and $-\mathbf{N}$, the claim follows from Proposition 6. $\square$

- To turn to the rationals, we need a new definition: that of Cartesian products.

- **Definition.** If X and Y are sets, we define the *Cartesian product* $X \times Y$ to be the set of pairs (x, y) where x is an element of X and y is an element of Y:

$$X \times Y := \{(x, y) : x \in X, y \in Y\}.$$

Two elements (x, y) and (x', y') in $X \times Y$ are said to be equal iff one has both $x = x'$ and $y = y'$.

- **Example.** If $X = \{0, 1, 3\}$, and $Y = \{4, 6\}$, then $X \times Y$ is the set $X \times Y = \{(0, 4), (0, 6), (1, 4), (1, 6), (3, 4), (3, 6)\}$. All six of these pairs are distinct; for instance, $(0, 4)$ and $(0, 6)$ are distinct because the second co-ordinates are unequal, even though the first co-ordinates are equal.

- For instance, $\mathbf{R}\times\mathbf{R}$ is the familiar Euclidean plane consisting of points (x, y) where x and y are both real numbers; we usually abbreviate $\mathbf{R}\times\mathbf{R}$ as $\mathbf{R}^2$.

- We will soon show that the set $\mathbf{N}\times\mathbf{N}$ is countable. We first need a preliminary lemma:

- **Lemma 8.** The set

$$A := \{(n, m) \in \mathbf{N}\times\mathbf{N} : 0 \leq m \leq n\}$$

  is countable.

- **Proof.** Define the sequence $a_0, a_1, a_2, \ldots$ recursively by setting $a_0 := 0$, and $a_{n++} := a_n + n{++}$ for all natural numbers n. Thus

$$a_0 = 0; a_1 = 0 + 1; a_2 = 0 + 1 + 2; a_3 = 0 + 1 + 2 + 3; \ldots.$$

  By induction one can show that a_n is increasing, i.e. that $a_n > a_m$ whenever $n > m$. (Why?)

  Now define the function $f : A \to \mathbf{N}$ by

$$f(n, m) := a_n + m.$$

  We claim that f is one-to-one. In other words, if (n, m) and (n', m') are any two distinct elements of A, then we claim that $f(n, m) \neq f(n', m')$.

- To prove this claim, let (n, m) and (n', m') be two distinct elements of A. There are three cases: $n' = n$, $n' > n$, and $n' < n$. First suppose that $n' = n$. Then we must have $m \neq m'$, otherwise (n, m) and (n', m') would not be distinct. Thus $a_n + m \neq a_n + m'$, and hence $f(n, m) \neq f(n', m')$, as desired.

- Now suppose that $n' > n$. Then $n' \geq n{++}$, and hence

$$f(n', m') = a_{n'} + m' \geq a_{n'} \geq a_{n++} = a_n + n{++}.$$

  But since $(n, m) \in A$, we have $m \leq n < n{++}$, and hence

$$f(n', m') \geq a_n + n{++} > a_n + m = f(n, m),$$

  and thus $f(n', m') \neq f(n, m)$.

- The case $n' < n$ is similar (just switch the roles of n and n' in the previous argument). Thus we have shown that f is one-to-one. Thus f is a bijection from A to $f(A)$, and thus A has equal cardinality with $f(A)$. But $f(A)$ is a subset of $\mathbf{N}$, and hence by Proposition 2 $f(A)$ is at most countable, hence A is at most countable. But, A is clearly not finite (why? Hint: if A was finite, then every subset of A would be finite, and in particular $\{(n, 0) : n \in \mathbf{N}\}$ would be finite, but this is clearly countably infinite, a contradiction). Thus, A must be countable. $\square$

- **Corollary 9.** The set $\mathbf{N} \times \mathbf{N}$ is countable.

- **Proof.** We already know that the set

$$A := \{(n, m) \in \mathbf{N} \times \mathbf{N} : 0 \leq m \leq n\}$$

is countable. This implies that the set

$$B := \{(n, m) \in \mathbf{N} \times \mathbf{N} : 0 \leq n \leq m\}$$

is also countable, since the map $f : A \to B$ given by $f(n, m) := (m, n)$ is a bijection from A to B (why?). But since $\mathbf{N} \times \mathbf{N}$ is the union of A and B (why?), the claim then follows from Proposition 6. $\square$

- **Corollary 10.** If X and Y are countable, then $X \times Y$ is countable.

- **Proof.** See Week 3 Homework. $\square$

- **Corollary 11.** The rationals $\mathbf{Q}$ are countable.

- **Proof.** We already know that the integers $\mathbf{Z}$ are countable, which implies that the non-zero integers $\mathbf{Z} - \{0\}$ are countable (why?). By Corollary 10, the set

$$\mathbf{Z} \times (\mathbf{Z} - \{0\}) = \{(a, b) : a, b \in \mathbf{Z}, b \neq 0\}$$

is thus countable. If one lets $f : \mathbf{Z} \times (\mathbf{Z} - \{0\}) \to \mathbf{Q}$ be the function $f(a, b) := a/b$ (note that f is well-defined since we prohibit b from being equal to 0), we thus see from Proposition 4 that $f(\mathbf{Z} \times (\mathbf{Z} - \{0\}))$ is at most countable. But we have $f(\mathbf{Z} \times (\mathbf{Z} - \{0\})) = \mathbf{Q}$ (why?

This is basically the definition of the rationals **Q**). Thus **Q** is at most countable. However, **Q** cannot be finite, since it contains the infinite set **N**. Thus **Q** is countable. ☐

- Because the rationals are countable, we know *in principle* that it is possible to arrange the rational numbers as a sequence:

$$\mathbf{Q} = \{a_0, a_1, a_2, a_3, \ldots\}$$

  such that every element of the sequence is different from every other element, and that the elements of the sequence exhaust **Q** (i.e. every rational number turns up as one of the elements a_n of the sequence). However, it is quite difficult (though not impossible) to actually try and come up with an explicit sequence $a_0, a_1, \ldots$ which does this. (Can you do it?)

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Uncountable sets

- **Definition** A set X is said to be *uncountable* iff it is neither countable nor finite (i.e. it is not at most countable).

- We have just shown that a lot of infinite sets are countable - even such sets as the rationals, for which it is not obvious how to arrange as a sequence. After such examples, one may begin to hope that other infinite sets, such as the real numbers, are also countable - after all, the real numbers are nothing more than (formal) limits of the rationals, and we've already shown the rationals are countable, so it seems plausible that the reals are also countable.

- It was thus a great shock when Georg Cantor showed in 1873 that certain sets - including the real numbers **R** are in fact uncountable - no matter how hard you try, you cannot arrange the real numbers **R** as a sequence $a_0, a_1, a_2, \ldots$. (Of course, the real numbers **R** can *contain* many infinite sequences, e.g. the sequence $0, 1, 2, 3, 4, \ldots$. However, what Cantor proved is that no such sequence can ever *exhaust* the real numbers; no matter what sequence of real numbers you choose, there will always be some real numbers that are not covered by that sequence).

- Cantor's proof requires some knowledge of the decimal system (see supplemental handout). We summarize the portions of the decimal system we will need here. To avoid possible confusion, we will refrain in this section from using the convention that ab denotes the product $a \times b$, because this will conflict with decimal notation (e.g. 34 would then equal 3×4).

- **Definition** A *digit* is any one of the numbers 0,1,2,3,4,5,6,7,8,9.

- **Definition** Let $a_1, a_2, a_3, \ldots$ be a sequence of rational numbers. We define the finite sums $\sum_{i=1}^n a_i$ recursively for every natural number n by the formulae

$$\sum_{i=1}^{0} a_i := 0; \quad \sum_{i=1}^{n++} a_i := (\sum_{i=1}^{n} a_i) + a_{n++} \text{ for every } n \in \mathbf{N}.$$

- Thus for instance $\sum_{i=1}^1 a_i = 0 + a_1 = a_1$; $\sum_{i=1}^2 a_i = a_1 + a_2$; and so forth. We will sometimes write $a_1 + a_2 + \ldots + a_n$ instead of $\sum_{i=1}^n a_i$. We can define more general sums $\sum_{i=m}^n a_i$, which start at some index m other than 1, for instance by the recursive definition

$$\sum_{i=m}^{n} a_i := 0 \text{ whenever } n < m; \sum_{i=m}^{n++} a_i := (\sum_{i=m}^{n} a_i) + a_{n++} \text{ whenever } n \geq m-1.$$

- **Definition** Let $a_1, a_2, a_3, \ldots$ be a sequence of digits. We then define the decimal $0.a_1a_2a_3\ldots$ to be the real number

$$0.a_1a_2a_3\ldots := \mathrm{LIM}_{n\to\infty} \sum_{i=1}^{n} a_n \times 10^{-n}.$$

- One can show that the sequence $\sum_{i=1}^n a_n \times 10^{-n}$ is a Cauchy sequence of rationals; see the supplemental handout on decimals.

- Thus, for instance, the decimal $0.111\ldots$ is the formal limit of the sequence

$$1 \times 10^{-1}, 1 \times 10^{-1} + 1 \times 10^{-2}, 1 \times 10^{-1} + 1 \times 10^{-2} + 1 \times 10^{-3}, \ldots$$

or more succinctly, the sequence

$$0.1, 0.11, 0.111, \ldots.$$

(Of course this latter statement is sort of circular, since I am using the notation of terminating decimals to describe a non-terminating decimal; but the first statement at least is rigorous and non-circular).

- One annoying feature of the decimal system is that some decimals are the same. For instance, the decimal $0.19999\ldots$ turns out to be exactly the same real number as $0.20000\ldots$ (see supplemental handout on decimals for more on this). However, we can avoid this problem if we restrict the set digits.

- **Definition** A *0-1 decimal* is a decimal of the form $0.a_1a_2a_3\ldots$ where all of the digits a_n are either 0 or 1.

- Thus, for instance $0.0000\ldots$, $0.111111\ldots$, and $0.10101010\ldots$ are 0-1 decimals, while $0.3333\ldots$ is not.

- The 0-1 decimals are all distinct:

- **Proposition 12.** Let $x := 0.a_1a_2a_3\ldots$ and $y := 0.b_1b_2b_3\ldots$ be 0-1 decimals, and suppose that the sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are not identical (i.e. we have $a_n \neq b_n$ for at least one positive integer n). Then we have $x \neq y$.

- **Proof (Optional).** Let X be the set of natural numbers n such that $a_n \neq b_n$. By hypothesis, X is non-empty, thus by the well-ordering principle the set X has a least element $N := \min(X)$. By definition of N, we thus have $a_N \neq b_N$, and also $a_n = b_n$ for all $1 \leq n < N$. Since $a_N \neq b_N$, we must have either $a_N < b_N$ or $b_N < a_N$; without loss of generality we may assume that $a_N < b_N$. Since both decimals are $0-1$ decimals, we must therefore have $a_N = 0$ and $b_N = 1$.

- Now let's try to estimate $y - x$. By definition of subtraction, we have

$$y - x = \mathrm{LIM}_{n\to\infty} \sum_{i=1}^n b_i \times 10^{-i} - \sum_{i=1}^n a_i \times 10^{-i}.$$

An easy induction shows that

$$\sum_{i=1}^{n} b_i \times 10^{-i} - \sum_{i=1}^{n} a_i \times 10^{-i} = \sum_{i=1}^{n} (b_i - a_i) \times 10^{-i}.$$

When $n < N$, then we have $b_i = a_i$ for all $1 \leq i \leq n$, and hence $\sum_{i=1}^{n} (b_i - a_i) \times 10^{-i} = 0$ (why? This is an easy induction). Now suppose that $n \geq N$. Then another induction shows that

$$\sum_{i=1}^{n} (b_i - a_i) \times 10^{-i} = (b_N - a_N) \times 10^{-N} + \sum_{i=N+1}^{n} (b_i - a_i) \times 10^{-i}.$$

Since $a_N = 0$ and $b_N = 1$, we thus have

$$\sum_{i=1}^{n} b_i \times 10^{-i} - \sum_{i=1}^{n} a_i \times 10^{-i} = 10^{-N} + \sum_{i=N+1}^{n} (b_i - a_i) \times 10^{-i}.$$

Now since b_i and a_i are both either 0 or 1, we have $b_i - a_i \geq -1$. Thus an easy induction shows that

$$\sum_{i=N+1}^{n} (b_i - a_i) \times 10^{-i} \geq - \sum_{i=N+1}^{n} 10^{-i}$$

and hence

$$\sum_{i=1}^{n} b_i \times 10^{-i} - \sum_{i=1}^{n} a_i \times 10^{-i} \geq 10^{-N} - \sum_{i=N+1}^{n} 10^{-i}.$$

On the other hand, an easy induction shows the geometric series formula

$$\sum_{i=N+1}^{n} 10^{-i} = (10^{-N} - 10^{-n})/9$$

and hence that

$$\sum_{i=1}^{n} b_i \times 10^{-i} - \sum_{i=1}^{n} a_i \times 10^{-i} \geq 10^{-N} - (10^{-N} - 10^{-n})/9 = \frac{8}{9} 10^{-N} + \frac{1}{9} 10^{-n} > \frac{8}{9} 10^{-N}.$$

(Note that we are now using the standard laws of algebra, without citing exactly where in the Week 1-2 notes this would come from; we will continue doing so for the rest of the course). Taking formal limits and using Corollary 22 from last week's notes (modified slightly to deal with the fact that the above inequality was only proven for $n \geq N$), we conclude that

$$y - x = \mathrm{LIM}_{n\to\infty} \sum_{i=1}^{n} b_i \times 10^{-i} - \sum_{i=1}^{n} a_i \times 10^{-i} > \frac{8}{9} 10^{-N} > 0$$

and hence in particular that $x \neq y$, as desired. $\square$

- Now we can come to Cantor's (rather non-trivial) argument.
- **Theorem 13.** Let X be the set of all $0-1$ decimals. Then X is uncountable.
- **Proof.** Suppose for contradiction that X was countable (it is pretty easy to show that X cannot be finite, for instance X contains the sequence $0.1, 0.01, 0.001, \ldots$ which is clearly countable). Then there is a bijection $f : \mathbf{Z}^+ \to X$ from the positive integers $\mathbf{Z}^+$ to the set X. Thus for every positive integer n we have a 0-1 decimal $f(n)$, which we write as
$$f(n) = 0.a_1^{(n)} a_2^{(n)} a_3^{(n)} \ldots$$
for some digits $a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots$ which are either 0 or 1. (We enclose the n in parentheses so that the superscript is not confused with exponentiation, we don't want a_j^n to be confused with $(a_j)^n$). Since f is assumed to be a bijection, then every 0-1 decimal can be written in the form $f(n)$ in exactly one way.
- Now we perform *Cantor's diagonal trick.* For each positive integer n, let b_n be the digit 0 if a_n^n is equal to 1, and let b_n be the digit 1 if a_n^n is equal to 0. (In other words, $b_n := 1 - a_n^n$). Thus $b_1, b_2, b_3, \ldots$ is a sequence of digits in 0 and 1 such that $b_n \neq a_n^n$ for every positive integer n.

  Let x be the decimal
$$x := 0.b_1 b_2 b_3 \ldots.$$

Clearly x is an 0-1 decimal, and thus lives in X. Since f is a bijection from $\mathbf{Z}^+$ to X, there must therefore exist a positive integer n such that $f(n) = x$, i.e.

$$0.b_1 b_2 b_3 \ldots = 0.a_1^n a_2^n a_3^n \ldots.$$

By Lemma 12, all the digits on both sides must match up, i.e. we must have $b_i = a_i^n$ for all positive integers i. In particular, we have $b_n = a_n^n$. But this contradicts the fact that $b_n \neq a_n^n$. Thus f could not have been a bijection. Thus X cannot be countable, and must therefore be uncountable (since it isn't finite). $\square$

- This diagonal trick is useful in problems concerning logic, set theory, or computability, and shows up in some important theorems such as Gödel's incompleteness theorem and Turing's halting theorem. But this is far beyond the scope of this course.

- **Corollary 14.** The real numbers $\mathbf{R}$ are uncountable.

- **Proof.** We know the real numbers are not finite, since they contain $\mathbf{N}$ (for instance). If the real numbers $\mathbf{R}$ were countable, then every subset of $\mathbf{R}$ would be at most countable by Corollary 3. But this would contradict Theorem 13, since the set X in that theorem is a subset of $\mathbf{R}$. $\square$

- The subject of countable and uncountable sets is a delicate one; there is much more to be said on this matter, but this is beyond the scope of this course and you must instead go to Math 112 for more details.

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Sequences of real numbers.

- In the Week 2 notes, we defined the real numbers as formal limits of rational (Cauchy) sequences, and we then defined various operations on the real numbers. However, we never really finished the job of constructing the real numbers, because we never got around to replacing formal limits $\mathrm{LIM}_{n\to\infty} a_n$ with actual limits $\lim_{n\to\infty} a_n$. In fact, we haven't defined limits at all yet. This will now be rectified.

- In the remainder of these notes, we will assume that a_n denotes a real number unless otherwise specified. The numbers n, m are also assumed to denote integers unless otherwise specified.

- We begin by repeating much of the machinery of ε-close sequences, etc. again - but this time, we do it for sequences of *real* numbers, not rational numbers. Thus this discussion will supercede what we did in the Week 2 notes.

- First, we define absolute value and distance for real numbers:

- **Definition.** Let x be a real number. Then we define the absolute value $|x|$ to equal x if x is positive, 0 if x is zero, and $-x$ if x is negative. Given two real numbers x and y, we define their distance $d(x, y)$ to be $d(x, y) := |x - y|$.

- Clearly this definition is consistent with the corresponding notions for rational numbers. Just as clearly, Proposition 1 from Week 2 notes works just as well for real numbers as it does for rationals (because the real numbers obey all the rules of algebra that the rationals do).

- **Definition.** Let $\varepsilon > 0$ be a real number. We say that two real numbers x, y are *ε-close* iff we have $d(y, x) \leq \varepsilon$.

- Again, it is clear that this definition of ε-close is consistent with the same definition we had for the rationals.

- Now let $(a_n)_{n=m}^\infty$ be a sequence of *real* numbers; i.e. we assign a real number a_n for every integer $n \geq m$. The starting index m is some integer; usually this will be 1, but in some cases we will start from some index other than 1. (The choice of label used to index this sequence is unimportant; we could use for instance $(a_k)_{k=m}^\infty$ and this would represent exactly the same sequence as $(a_n)_{n=m}^\infty$). We can define the notion of a Cauchy sequence in the same manner as before:

- **Definition.** Let $\varepsilon > 0$ be a real number. A sequence $(a_n)_{n=N}^\infty$ of real numbers starting at some integer index N is said to be *ε-steady* iff a_j and a_k are ε-close for every $j, k \geq N$. A sequence $(a_n)_{n=m}^\infty$ starting at some integer index m is said to be *eventually ε-steady* iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε-steady. We say that $(a_n)_{n=m}^{\infty}$ is a *Cauchy sequence* iff it is eventually ε-steady for every $\varepsilon > 0$.

- These definitions are consistent with the rational definitions, although the one for Cauchy sequences takes a little bit of care:

- **Proposition 15.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of rationals starting at some integer index m. Then $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence (in the sense of the Week 2 definition for rationals) if and only if it is a Cauchy sequence (in the sense of the current definition for reals).

- **Proof.** Suppose first that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence using the current definition; then it is eventually ε-steady for every *real* $\varepsilon > 0$. In particular, it is eventually ε-steady for every *rational* $\varepsilon > 0$, which makes it a Cauchy sequence in the sense of the Week 2 notes.

- Now suppose that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence using the Week 2 definition; then it is eventually ε-steady for every *rational* $\varepsilon > 0$. If $\varepsilon > 0$ is a real number, then there exists a *rational* $\varepsilon' > 0$ which is smaller than ε, by Proposition 25 of Week 2 notes. Since ε' is rational, we know that $(a_n)_{n=m}^{\infty}$ is eventually ε'-steady; since $\varepsilon' < \varepsilon$, this implies that $(a_n)_{n=m}^{\infty}$ is eventually ε-steady. Since ε is an arbitrary positive real number, we thus see that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence using the current definition. □

- Because of this proposition, we will no longer care about the distinction between this definition of a Cauchy sequence and the previous one.

- Now we talk about what it means for a sequence of real numbers to converge to some limit L.

- **Definition.** Let $\varepsilon > 0$ be a real number, and let L be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers is said to be *ε-close to L* iff a_n is ε-close to L for every $n \geq N$, i.e. we have $|a_n - L| \leq \varepsilon$ for every $n \geq N$. We say that a sequence $(a_n)_{n=m}^{\infty}$ is *eventually ε-close to L* iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε-close to L. We say that a sequence $(a_n)_{n=m}^{\infty}$ *converges to L* iff it is eventually ε-close to L for every real $\varepsilon > 0$.

- **Examples**. The sequence

$$0.9, 0.99, 0.999, 0.9999, \ldots$$

is 0.1-close to 1, but is not 0.01-close to 1, because of the first element of the sequence. However, it is eventually 0.01-close to 1. In fact, for every real $\varepsilon > 0$, this sequence is eventually ε-close to 1, hence is convergent to 1.

- One can rewrite the definition of convergence as follows: the sequence $(a_n)_{n=m}^\infty$ converges to L if, given any real $\varepsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$. (Why is this the same definition as the one given above?)

- **Proposition 16.** Let $(a_n)_{n=m}^\infty$ be a real sequence starting at some integer index m, and let $L \neq L'$ be two distinct real numbers. Then it is not possible for $(a_n)_{n=m}^\infty$ to converge to L while also converging to L'.

- **Proof.** Suppose for contradiction that $(a_n)_{n=m}^\infty$ was converging both to L and to L'. Let $\varepsilon = |L - L'|/3$; note that ε is positive since $L \neq L'$. Since $(a_n)_{n=m}^\infty$ is converging to L, we know that $(a_n)_{n=m}^\infty$ is eventually ε-close to L; thus there is an $N \geq m$ such that $d(a_n, L) \leq \varepsilon$ for all $n \geq N$. Similarly, there is an $M \geq m$ such that $d(a_n, L') \leq \varepsilon$ for all $n \geq M$. In particular, if we set $n := \max(N, M)$, then we have $d(a_n, L) \leq \varepsilon$ and $d(a_n, L') \leq \varepsilon$, hence by the triangle inequality $d(L, L') \leq 2\varepsilon = 2|L - L'|/3$. But then we have $|L - L'| \leq 2|L - L'|/3$, which contradicts the fact that $|L - L'| > 0$. Thus it is not possible to converge to both L and L'. $\square$

- **Definition** If a sequence $(a_n)_{n=m}^\infty$ is converging to some real number L, we say that the sequence $(a_n)_{n=m}^\infty$ is *convergent* and that its limit is L; we write
$$L = \lim_{n\to\infty} a_n$$
to denote this fact. If a sequence $(a_n)_{n=m}^\infty$ is not converging to any real number L, we say that the sequence $(a_n)_{n=m}^\infty$ is *divergent* and we leave $\lim_{n\to\infty} a_n$ undefined.

- Note that Proposition 16 ensures that a sequence can have at most one limit. Thus, if the limit exists, it is a single real number, otherwise it is undefined.

- The notation $\lim_{n\to\infty} a_n$ does not give any indication about the starting index m of the sequence, but the starting index is irrelevant: if $(a_n)_{n=m}^{\infty}$ is convergent to some limit c, then $(a_n)_{n=m'}^{\infty}$ is also convergent to the same limit c for any other m' (provided of course that all the elements of the sequence are defined and are real numbers). (Why?) Thus in the rest of this discussion we shall not be too careful as to where these sequences start, as we shall be mostly focused on their limits.

- In a similar spirit, one can check that if $(a_n)_{n=m}^{\infty}$ is convergent to some limit c, then $(a_{n+k})_{n=m-k}^{\infty}$ is also convergent to c for any integer k (can you prove this rigorously?).

- We sometimes use the phrase "$a_n \to x$ as $n \to \infty$" as an alternate way of writing the statement "$(a_n)_{n=m}^{\infty}$ converges to x". Bear in mind, though, that the individual statements $a_n \to x$ and $n \to \infty$ do not have any rigorous meaning; this phrase is just a convention, though of course a very suggestive one.

- Finally, the exact choice of letter used to denote the index (in this case n) is irrelevant: the phrase $\lim_{n\to\infty} a_n$ has exactly the same meaning as $\lim_{k\to\infty} a_k$, for instance. Sometimes it will be convenient to change the label of the index to avoid conflicts of notation (for instance, we might want to change n to k because n is simultaneously being used for some other purpose, and we want to reduce confusion).

- As an example of a limit, we present

- **Proposition 17**. We have $\lim_{n\to\infty} 1/n = 0$.

- **Proof.** We have to show that the sequence $(a_n)_{n=1}^{\infty}$ converges to 0, where $a_n := 1/n$. In other words, for every $\varepsilon > 0$, we need to show that the sequence $(a_n)_{n=1}^{\infty}$ is eventually ε-close to 0. So, let $\varepsilon > 0$ be an arbitrary real number. We have to find an N such that $|a_n - 0| \leq \varepsilon$ for every $n \geq N$. But if $n \geq N$, then

$$|a_n - 0| = |1/n - 0| = 1/n \leq 1/N.$$

Thus, if we pick $N > 1/\varepsilon$ (which we can do by the Archimedean principle), then $1/N < \varepsilon$, and so $(a_n)_{n=N}^\infty$ is ε-close to 0. Thus $(a_n)_{n=1}^\infty$ is eventually ε-close to 0. Since ε was arbitrary, $(a_n)_{n=1}^\infty$ converges to 0.$\square$

- Now we connect the notions of convergent sequences and Cauchy sequences.
- **Proposition 18.** Suppose that $(a_n)_{n=m}^\infty$ is a convergent sequence of real numbers. Then it is also a Cauchy sequence.
- **Proof.** See Week 4 homework. $\square$
- **Example**. The sequence $1, -1, 1, -1, 1, -1, \ldots$ is not a Cauchy sequence (because it is not eventually 1-steady), and is hence not a convergent sequence, by Proposition 18.
- There is a converse to this Proposition (Theorem 30), but we will wait a bit before proving it.
- Now we show that formal limits can be superceded by actual limits, just as formal subtraction was superceded by actual subtraction and formal division superceded by actual division.
- **Proposition 19.** Let $(a_n)_{n=1}^\infty$ be a Cauchy sequence of rational numbers. Then $(a_n)_{n=1}^\infty$ converges to $\mathrm{LIM}_{n\to\infty} a_n$, i.e.

$$\mathrm{LIM}_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

- **Proof.** See Week 4 homework. $\square$
- **Definition** A sequence $(a_n)_{n=m}^\infty$ of real numbers is *bounded by M* iff we have $|a_n| \leq M$ for all $n \geq m$. We say that $(a_n)_{n=m}^\infty$ is *bounded* iff it is bounded by M for some real number $M > 0$.
- By arguing as in Proposition 15 we can see that this definition of a bounded sequence is consistent with the definition in Week 2 notes (which was just the same, except that everything was assumed to be rational instead of real).

- Recall from Lemma 9 of Week 2 notes that every Cauchy sequence of rational numbers is bounded. An inspection of the proof of that Lemma shows that the same argument works for real numbers; every Cauchy sequence of real numbers is bounded. In particular, from Proposition 18 we see have

- **Corollary 20.** Every convergent sequence is bounded.

- **Example.** The sequence $1, 2, 3, 4, 5, \ldots$ is not bounded, and hence is not convergent.

- We now can prove the usual limit laws.

- **Theorem 21 (Limit Laws).** Let $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ be convergent sequences, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$.

- (a) The sequence $(a_n + b_n)_{n=m}^\infty$ converges to $x + y$; in other words,

$$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

- (b) The sequence $(a_n b_n)_{n=m}^\infty$ converges to xy; in other words,

$$\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

- (c) For any real number c, the sequence $(ca_n)_{n=m}^\infty$ converges to cx, in other words

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n.$$

- (d) The sequence $(a_n - b_n)_{n=m}^\infty$ converges to $x - y$; in other words,

$$\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

- (e) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^\infty$ converges to y^{-1}; in other words,

$$\lim_{n\to\infty} b_n^{-1} = (\lim_{n\to\infty} b_n)^{-1}.$$

- (f) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y; in other words,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

- **Proof.** See Week 4 Homework. $\square$

- Note that Theorem 21(f) doesn't work when the limit of the denominator is 0. To address that problem requires L'Hopital's rule, which we will obtain later in this course.

* * * * *

The Extended Real number system

- There are some sequences which do not converge to any real number, but instead seem to be wanting to converge to $+\infty$ or $-\infty$. For instance, it seems intuitive that the sequence

$$1, 2, 3, 4, 5, \ldots$$

should be converging to $+\infty$, while

$$-1, -2, -3, -4, -5, \ldots$$

should be converging to $-\infty$. Meanwhile, the sequence

$$1, -1, 1, -1, 1, -1, \ldots$$

does not seem to be converging to much of anything (although we shall see later that it does have $+1$ and -1 as "limit points" - see below). Similarly the sequence

$$1, -2, 3, -4, 5, -6, \ldots$$

does not converge to any real number, and also does not appear to be converging to $+\infty$ or converging to $-\infty$.

- To make this precise we need to talk about something called the *extended real number system.*

- **Definition** The *extended real number system* $\mathbf{R}^*$ is the real line $\mathbf{R}$ with two additional elements attached, called $+\infty$ and $-\infty$. These elements are distinct from each other and also distinct from every real number. An extended real number x is called *finite* iff it is a real number, and *infinite* iff it is equal to $+\infty$ or $-\infty$. (This definition is not directly related to the notion of finite and infinite sets discussed earlier, though it is of course similar in spirit).

- These new symbols, $+\infty$ and $-\infty$, at present do not have much meaning, since we have no operations to manipulate them (other than equality $=$ and inequality $\neq$). Now we place a few operations on the extended real number system.

- **Definition** The operation of negation $x \mapsto -x$ on $\mathbf{R}$, we now extend to $\mathbf{R}^*$ by defining $-(+\infty) := -\infty$ and $-(-\infty) := +\infty$.

- Thus every extended real number x has a negation, and $-(-x)$ is always equal to x.

- **Definition** Let x and y be extended real numbers. We say that $x \leq y$, i.e. x is less than or equal to y, iff one of the following three statements is true:

- (a) x and y are real numbers, and $x \leq y$ as real numbers.

- (b) $y = +\infty$.

- (c) $x = -\infty$.

- We say that $x < y$ if we have $x \leq y$ and $x \neq y$. We sometimes write $x < y$ instead as $y > x$, and $x \leq y$ sometimes as $y \geq x$.

- Thus for instance, $3 \leq 5$, $3 < +\infty$, and $-\infty < +\infty$, but $3 \not\leq -\infty$.

- Some basic properties of order and negation on the extended real number system:

- **Proposition 22.** Let x, y, z be extended real numbers. Then the following statements are true:

- (a) We have $x \leq x$.

- (b) If $x \leq y$ and $y \leq x$, then $x = y$.
- (c) If $x \leq y$ and $y \leq z$, then $x \leq z$.
- (d) If $x \leq y$, then $-y \leq -x$.
- **Proof.** The number x is either a real number, or $+\infty$, or $-\infty$. Similarly for y and z. Thus we can break into a large number of cases, and check each case by hand (using Proposition 19 from Week 2 notes as necessary). This is rather tedious and will not be done here (and I won't inflict it on you as homework either). $\square$
- One could also introduce other operations on the extended real number system, such as addition, multiplication, etc. However, this is somewhat dangerous as these operations will almost certainly fail to obey the familiar rules of algebra. For instance, to define addition it seems reasonable (given one's intuitive notion of infinity) to set $+\infty+5 = +\infty$ and $+\infty + 3 = +\infty$, but then this implies that $+\infty + 5 = +\infty + 3$, while $5 \neq 3$. So things like the cancellation law begin to break down once we try to operate involving infinity. To avoid these issues we shall simply not define any arithmetic operations on the extended real number system other than negation and order. (You can read pages 11-12 of the textbook for some ways to define those arithmetic operations on $\mathbf{R}^*$).
- Remember that we defined the notion of *supremum* or *least upper bound* of a set E of reals; this gave an extended real number $\sup(E)$, which was either finite or infinite. We now extend this notion slightly.
- **Definition** Let E be a subset of $\mathbf{R}^*$. Then we define the *supremum* $\sup(E)$ or *least upper bound* of E by the following rule.
- (a) If E is contained in $\mathbf{R}$ (i.e. $+\infty$ and $-\infty$ are not elements of E), then we let $\sup(E)$ be as defined in Week 2 notes.
- (b) If E contains $+\infty$, then we set $\sup(E) := +\infty$.
- (c) If E does not contain $+\infty$ but does contain $-\infty$, then we set $\sup(E) := \sup(E - \{-\infty\})$ (which is a subset of $\mathbf{R}$ and so falls under case (a)).

- We also define the *infimum* $\inf(E)$ of E (also known as the *greatest lower bound* of E by the formula

$$\inf(E) := -\sup(-E)$$

where $-E$ is the set $-E := \{-x : x \in E\}$.

- (A Latin note: supremum means "highest" and infimum means "lowest", and the plurals are suprema and infima. Supremum is to superior, and infimum to inferior, as maximum is to major, and minimum to minor. The root words are "super", which means "above", and "infer", which means "below" (this usage only survives in a few rare English words such as "infernal", with the Latin prefix "sub" having mostly replaced "infer" in English)).

- **Example** Let E be the negative integers, together with $-\infty$:

$$E = \{-1, -2, -3, -4, \ldots\} \cup \{-\infty\}.$$

Then $\sup(E) = \sup(E - \{-\infty\}) = -1$, while $\inf(E) = -\sup(-E) = -(+\infty) = -\infty$.

- **Example** The set $\{0.9, 0.99, 0.999, 0.9999, \ldots\}$ has infimum 0.9 and supremum 1. Note that in this case the supremum does not actually belong to the set, but it is in some sense "touching it" from the right.

- **Example** The set $\{1, 2, 3, 4, 5 \ldots\}$ has infimum 1 and supremum $+\infty$.

- **Example.** Let E be the empty set. Then $\sup(E) = -\infty$ and $\inf(E) = +\infty$ (why?). This is the only case in which the supremum can be less than the infimum (why?).

- One can intuitively think of the supremum of E as follows. Imagine the real line with $+\infty$ somehow on the far right, and $-\infty$ on the far left. Imagine a piston at $+\infty$ moving leftward until it is stopped by the presence of a set E; the location where it stops is the supremum of E. Similarly if one imagines a piston at $-\infty$ moving rightward until it is stopped by the presence of E, the location where it stops is the infimum of E. In the case when E is the empty set, the pistons pass through each other, the supremum landing at $-\infty$ and the infimum landing at $+\infty$.

- The following theorem justifies the terminology "least upper bound" and "greatest lower bound":
- **Theorem 23.** Let E be a subset of $\mathbf{R}^*$. Then the following statements are true.
- (a) For every $x \in E$ we have $x \leq \sup(E)$ and $x \geq \inf(E)$.
- (b) Suppose that M is an upper bound for E, i.e. $x \leq M$ for all $x \in E$. Then we have $\sup(E) \leq M$.
- (c) Suppose that M is a lower bound for E, i.e. $x \geq M$ for all $x \in E$. Then we have $\inf(E) \geq M$.
- **Proof.** See Week 4 Homework. $\square$

* * * * *

Suprema and Infima of sequences

- We now define the supremum and infimum of a sequence.
- **Definition.** Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers. Then we define $\sup(a_n)_{n=m}^\infty$ to be the supremum of the set $\{a_n : n \geq m\}$, and $\inf(a_n)_{n=m}^\infty$ to the infimum of the same set $\{a_n : n \geq m\}$.
- **Example.** Let $a_n := (-1)^n$; thus $(a_n)_{n=1}^\infty$ is the sequence $-1, 1, -1, 1, \ldots$. Then the set $\{a_n : n \geq 1\}$ is just the two-element set $\{-1, 1\}$, and hence $\sup(a_n)_{n=1}^\infty$ is equal to 1. Similarly $\inf(a_n)_{n=1}^\infty$ is equal to -1.
- **Example.** Let $a_n := 1/n$; thus $(a_n)_{n=1}^\infty$ is the sequence $1, 1/2, 1/3, \ldots$. Then the set $\{a_n : n \geq 1\}$ is the countable set $\{1, 1/2, 1/3, 1/4, \ldots\}$. Thus $\sup(a_n)_{n=1}^\infty = 1$ and $\inf(a_n)_{n=1}^\infty = 0$ (can you prove these two statements rigorously?). Notice here that the infimum of the sequence is not actually a member of the sequence, though it becomes very close to the sequence eventually. (So it is a little inaccurate to think of the supremum and infimum as the "largest element of the sequence" and "smallest element of the sequence" respectively).
- **Example.** Let $a_n := n$; thus $(a_n)_{n=1}^\infty$ is the sequence $1, 2, 3, 4, \ldots$. Then the set $\{a_n : n \geq 1\}$ is just the positive integers $\{1, 2, 3, 4, \ldots\}$. Then $\sup(a_n)_{n=1}^\infty = +\infty$ and $\inf(a_n)_{n=1}^\infty = 1$.

- As the last example shows, it is possible for the supremum or infimum of a sequence to be $+\infty$ or $-\infty$. However, if a sequence $(a_n)_{n=m}^{\infty}$ is bounded, say bounded by M, then all the elements a_n of the sequence lie between $-M$ and M, so that the set $\{a_n : n \geq m\}$ has M as an upper bound and $-M$ as a lower bound. Since this set is clearly non-empty, we can thus conclude that its supremum and infimum are real numbers (i.e. not $+\infty$ and $-\infty$, and also lie between $-M$ and M).

- **Proposition 24.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be the extended real number $x := \sup(a_n)_{n=m}^{\infty}$. Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in \mathbf{R}^*$ is an upper bound for a_n (i.e. $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which $y < a_n \leq x$.

- **Proof.** See Week 4 Homework. $\square$

- There is a corresponding Proposition for infima, but with all the references to order reversed, e.g. all upper bounds should now be lower bounds, etc. The proof is exactly the same.

- Now we give an application of these concepts of supremum and infimum. In the previous section we saw that all convergent sequences are bounded. It is natural to ask whether the converse is true: are all bounded sequences convergent? The answer is no; for instance, the sequence $1, -1, 1, -1, \ldots$ is bounded, but not Cauchy and hence not convergent. However, if we make the sequence both bounded and *monotone* (i.e. increasing or decreasing), then it is true that it must converge:

- **Proposition 25.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite upper bound $M \in \mathbf{R}$, and which is also increasing (i.e. $a_{n++} \geq a_n$ for all $n \geq m$). Then $(a_n)_{n=m}^{\infty}$ is convergent, and in fact

$$\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty} \leq M.$$

- **Proof.** See Week 4 Homework. $\square$

- One can similarly prove that if a sequence $(a_n)_{n=m}^\infty$ is bounded and decreasing (i.e. $a_{n++} \leq a_n$), then it is convergent, and that the limit is equal to the infimum.

- A sequence is said to be *monotone* if it is either increasing or decreasing. From Proposition 25 and Corollary 20 we see that a monotone sequence converges if and only if it is bounded.

- **Example.** The sequence $3, 3.1, 3.14, 3.141, 3.1415, \ldots$ is increasing, and is bounded above by 4. Hence by Proposition 23 it must have a limit, which is a real number less than or equal to 4.

- We can use Proposition 25 to compute certain limits. Here is an example:

- **Claim.** Let $0 < x < 1$. Then we have $\lim_{n\to\infty} x^n = 0$.

- **Proof.** Since $0 < x < 1$, one can show that the sequence $(x^n)_{n=1}^\infty$ is decreasing (why?). On the other hand, the sequence $(x^n)_{n=1}^\infty$ has a lower bound of 0. Thus by Proposition 25 (for infima instead of suprema) the sequence $(x^n)_{n=1}^\infty$ converges to some limit L. Since $x^{n+1} = x \times x^n$, we thus see from limit laws that $(x^{n+1})_{n=1}^\infty$ converges to xL. But the sequence $(x^{n+1})_{n=1}^\infty$ is just the sequence $(x^n)_{n=2}^\infty$ shifted by one, and so they must have the same limits (why?). So $xL = L$. Since $x \neq 1$, we can solve for L to obtain $L = 0$. Thus $(x^n)_{n=1}^\infty$ converges to 0. $\square$

- Note how this proof does not work when $x > 1$ (why?). Indeed one can show that x^n is divergent in this case (why? Prove by contradiction and use the identity $(1/x)^n x^n = 1$ and limit laws).

* * * * *

Limsup, Liminf, and limit points

- Consider the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots.$$

- If one plots this sequence, then one sees (informally, of course) that this sequence does not converge; half the time the sequence is getting close to 1, and half the time the sequence is getting close to -1, but it is not converging to either of them (for instance, it never gets eventually 1/2-close to 1, and never gets eventually 1/2-close to -1). However, even though -1 and +1 are not quite limits of this sequence, it does seem that in some vague way they "want" to be limits. To make this notion precise we introduce the notion of a *limit point*.

- **Definition** Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers, let x be a real number, and let $\varepsilon > 0$ be a real number. We say that x is *ε-adherent* to $(a_n)_{n=m}^\infty$ iff there exists an $n \geq m$ such that a_n is ε-close to x. We say that x is *continually ε-adherent* to $(a_n)_{n=m}^\infty$ iff it is ε-adherent to $(a_n)_{n=N}^\infty$ for every $N \geq m$. We say that x is a *limit point* or *adherent point* of $(a_n)_{n=m}^\infty$ iff it is continually ε-adherent to $(a_n)_{n=m}^\infty$ for every $\varepsilon > 0$.

- (The verb "to adhere" means much the same as "to stick to"; hence the term "adhesive".) Note that limit points are only defined for finite real numbers; it is possible to make rigorous also the concept of $+\infty$ or $-\infty$ being a limit point, but we will not do so here.

- **Example.** Let $(a_n)_{n=1}^\infty$ denote the sequence

$$0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots.$$

The number 0.8 is 0.1-adherent to this sequence, since 0.8 is 0.1-close to 0.9, which is a member of this sequence. However, it is not *continually* 0.1-adherent to this sequence, since once one discards the first element of this sequence there is no member of the sequence to be 0.1-close to. In particular, 0.8 is not a limit point of this sequence. On the other hand, the number 1 is 0.1-adherent to this sequence, and in fact is continually 0.1-adherent to this sequence, since no matter how many initial members of the sequence one discards, there is still something for 1 to be 0.1-close to. In fact, it is continually ε-adherent for every ε, and is hence a limit point of this sequence.

- **Example.** Now consider the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots.$$

The number 1 is 0.1-adherent to this sequence; in fact it is continually 0.1-adherent to this sequence, because no matter how many elements of the sequence one discards, there are some elements of the sequence that 1 is 0.1-close to. (Note that one does not need *all* the elements to be 0.1-close to 1, just some; thus 0.1-adherent is weaker than 0.1-close, and continually 0.1-adherent is a different notion from eventually 0.1-close). In fact, for every $\varepsilon > 0$, the number 1 is continually ε-adherent to this sequence, and is thus a limit point of this sequence. Similarly -1 is a limit point of this sequence; however 0 (say) is not a limit point of this sequence, since it is not continually 0.1-adherent to it.

- Unwrapping all the definitions, we see that x is a limit point of $(a_n)_{n=m}^\infty$ if, for every $\varepsilon > 0$ and every $N \geq m$, there exists an $n \geq N$ such that $|a_n - x| \leq \varepsilon$. (Why is this the same definition?).

- Limits are of course a special case of limit points:

- **Proposition 26.** Let $(a_n)_{n=m}^\infty$ be a sequence which converges to a real number c. Then c is a limit point of $(a_n)_{n=m}^\infty$, and in fact it is the only limit point of $(a_n)_{n=m}^\infty$.

- **Proof.** See Week 4 homework. □

- Now we will look at two special types of limit points: the limit superior (lim sup) and limit inferior (lim inf).

- Suppose that $(a_n)_{n=m}^\infty$ is a sequence. We define a new sequence $(a_N^+)_{N=m}^\infty$ by the formula

$$a_N^+ := \sup(a_n)_{n=N}^\infty.$$

Thus a_N^+ is the supremum of a_N, and all the elements in the sequence after a_N. We then define the *limit superior* of the sequence $(a_n)_{n=m}^\infty$, denoted $\limsup_{n\to\infty} a_n$, by the formula

$$\limsup_{n\to\infty} a_n := \inf(a_N^+)_{N=m}^\infty.$$

Similarly, we can define

$$a_N^- := \inf(a_n)_{n=N}^\infty$$

and define the *limit inferior* of the sequence $(a_n)_{n=m}^\infty$, denoted $\liminf_{n\to\infty} a_n$, by the formula

$$\liminf_{n\to\infty} a_n := \sup(a_N^-)_{N=m}^\infty.$$

- **Example.** Let $a_1, a_2, a_3, \ldots$ denote the sequence

  $$1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots.$$

  Then $a_1^+, a_2^+, a_3^+, \ldots$ is the sequence

  $$1.1, 1.001, 1.001, 1.00001, 1.00001, \ldots$$

  (why?), and its infimum is 1. Hence the limit superior of this sequence is 1. Similarly, $a_1^-, a_2^-, a_3^-, \ldots$ is the sequence

  $$-1.01, -1.01, -1.0001, -1.0001, -1.000001, \ldots$$

  (why?), and the supremum of this sequence is -1. Hence the limit inferior of this sequence is -1. One should compare this with the supremum and infimum of the sequence, which are 1.1 and -1.01 respectively.

- **Example.** Let $a_1, a_2, a_3, \ldots$ denote the sequence

  $$1, -2, 3, -4, 5, -6, 7, -8, \ldots$$

  Then $a_1^+, a_2^+, \ldots$ is the sequence

  $$+\infty, +\infty, +\infty, +\infty, \ldots$$

  (why?) and so the limit superior is $+\infty$. Similarly, $a_1^-, a_2^-, \ldots$ is the sequence

  $$-\infty, -\infty, -\infty, -\infty, \ldots$$

  and so the limit inferior is $-\infty$.

- **Example.** Let $a_1, a_2, a_3, \ldots$ denote the sequence

  $$1, -1/2, 1/3, -1/4, 1/5, -1/6, \ldots$$

  Then $a_1^+, a_2^+, \ldots$ is the sequence

  $$1, 1/3, 1/3, 1/5, 1/5, 1/7, \ldots$$

which has an infimum of 0 (why?), so the limit superior is 0. Similarly, $a_1^-, a_2^-, \ldots$ is the sequence

$$-1/2, -1/2, -1/4, -1/4, -1/6, -1/6$$

which has a supremum of 0. So the limit inferior is also 0.

- **Example.** Let $a_1, a_2, a_3, \ldots$ denote the sequence

  $$1, 2, 3, 4, 5, 6, \ldots$$

  Then $a_1^+, a_2^+, \ldots$ is the sequence

  $$+\infty, +\infty, +\infty, \ldots$$

  so the limit superior is $+\infty$. Similarly, $a_1^-, a_2^-, \ldots$ is the sequence

  $$1, 2, 3, 4, 5, \ldots$$

  which has a supremum of $+\infty$. So the limit inferior is also $+\infty$.

- Some authors use $\overline{\lim}_{n\to\infty} a_n$ instead of $\limsup_{n\to\infty} a_n$, and $\underline{\lim}_{n\to\infty} a_n$ instead of $\liminf_{n\to\infty} a_n$. Note that the starting index m of the sequence is irrelevant; if one removes the first few elements of the sequence, e.g. using instead the sequence $(a_n)_{n=M}^\infty$ for some $M > m$, then this does not affect either the limit superior or limit inferior (why?). Similarly, shifting the sequence by replacing a_n with a_{n+k} does not affect the limit superior or limit inferior (why?).

- Returning to the piston analogy, imagine a piston at $+\infty$ moving leftward until it is stopped by the presence of the sequence $a_1, a_2, \ldots$. The place it will stop is the supremum of $a_1, a_2, a_3, \ldots$, which in our new notation is a_1^+. Now let us remove the first element a_1 from the sequence; this may cause our piston to slip leftward, to a new point a_2^+ (though in many cases the piston will not move and a_2^+ will just be the same as a_1^+). Then we remove the second element a_2, causing the piston to slip a little more. If we keep doing this the piston will keep slipping, but there will be some point where it cannot go any further, and this is the limit superior of the sequence. A similar analogy can describe the limit inferior of the sequence.

- We now describe some basic properties of limit superior and limit inferior.

- **Proposition 27.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence (thus both L^+ and L^- are extended real numbers).

- (a) For every $x > L^+$, there exists an $N \geq m$ such that $a_n < x$ for all $n \geq N$. (In other words, for every $x > L^+$, the sequence $(a_n)_{n=m}^{\infty}$ is eventually less than x). Similarly, for every $y < L^-$ there exists an $N \geq m$ such that $a_n > y$ for all $n \geq N$.

- (b) For every $x < L^+$, and every $N \geq m$, there exists an $n \geq N$ such that $a_n > x$. (In other words, for every $x < L^+$, the sequence $(a_n)_{n=m}^{\infty}$ is continually greater than x). Similarly, for every $y > L^-$ and every $N \geq m$, there exists an $n \geq N$ such that $a_n < y$.

- (c) We have $\inf(a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^{\infty}$.

- (d) If c is any limit point of $(a_n)_{n=m}^{\infty}$, then we have $L^- \leq c \leq L^+$.

- (e) If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$. Similarly, if L^- is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$.

- (f) Let c be a real number. If $(a_n)_{n=m}^{\infty}$ converges to c, then we must have $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^{\infty}$ converges to c.

- **Proof.** We shall prove (a) and (b), and leave (cdef) to the exercises. Suppose first that $x > L^+$. Then by definition of L^+, we have $x > \inf(a_N^+)_{N=m}^{\infty}$. By Proposition 24, there must then exist an integer $N \geq m$ such that $x > a_N^+$. By definition of a_N^+, this means that $x > \sup(a_n)_{n=N}^{\infty}$. Thus by Proposition 24 again, we have $x > a_n$ for all $n \geq N$, as desired. This proves the first part of (a); the second part of (a) is proven similarly.

  Now we prove (b). Suppose that $x < L^+$. Then we have $x < \inf(a_N^+)_{N=m}^{\infty}$. If we fix any $N \geq m$, then by Proposition 24, we thus have $x < a_N^+$. By definition of a_N^+, this means that $x < \sup(a_n)_{n=N}^{\infty}$. By Proposition 24 again, there must thus exist $n \geq N$ such that $a_n > x$, as desired. This proves the first part of (b), the second part of (b) is proven similarly.

The proofs of (c), (d), (e), (f) are left to the Week 4 homework. $\square$

- Proposition 27(cd) says, in particular, that L^+ is the largest limit point of $(a_n)_{n=m}^\infty$, and L^- is the smallest limit point (providing that L^+ and L_- are finite. Proposition 27 (f) then says that if L^+ and L^- co-incide (so there is only one limit point), then the sequence in fact converges. This gives a way to test if a sequence converges: compute its limit superior and limit inferior, and see if they are equal.

- We now give a basic comparison property of limit superior and limit inferior.

- **Lemma 28 (Comparison principle).** Let $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ be two sequences of real numbers, and suppose that $a_n \leq b_n$ for all $n \geq m$. Then we have
$$\sup(a_n)_{n=m}^\infty \leq \sup(b_n)_{n=m}^\infty$$
and
$$\inf(a_n)_{n=m}^\infty \leq \inf(b_n)_{n=m}^\infty$$
and
$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$$
and
$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

- **Proof.** See Week 4 Homework. $\square$

- **Corollary 29 (Squeeze test).** Let $(a_n)_{n=m}^\infty$, $(b_n)_{n=m}^\infty$, and $(c_n)_{n=m}^\infty$ be sequences of real numbers such that
$$a_n \leq b_n \leq c_n$$
for all $n \geq M$. Suppose also that $(a_n)_{n=m}^\infty$ and $(c_n)_{n=m}^\infty$ both converge to the same limit L. Then $(b_n)_{n=m}^\infty$ is also convergent to L.

- **Proof.** See Week 4 Homework. $\square$

- **Example.** We already know (see Proposition 17) that $\lim_{n\to\infty} 1/n = 0$. By the limit laws, this also implies that $\lim_{n\to\infty} 2/n = 0$ and $\lim_{n\to\infty} -2/n = 0$. The squeeze test then shows that any sequence $(b_n)_{n=1}^{\infty}$ for which

$$-2/n \leq b_n \leq 2/n \text{ for all } n \geq 1$$

is convergent to 0. For instance, we can use this to show that the sequence $(-1)^n/n + 1/n^2$ converges to zero, or that 2^{-n} converges to zero (note one can use induction to show that $0 \leq 2^{-n} \leq 1/n$ for all $n \geq 1$).

- This squeeze test, combined with the limit laws and the principle that monotone bounded sequences always have limits, allows to compute a large number of limits. We give some examples in the next set of notes.

- Finally, we can give the converse to Proposition 18:

- **Theorem 30.** Every Cauchy sequence $(a_n)_{n=1}^{\infty}$ of real numbers is also convergent.

- Note that while this is very similar in spirit to Proposition 19, it is a bit more general (Proposition 19 refers to Cauchy sequences of rationals, not real numbers).

- **Proof.** We know from Corollary 20 that the sequence $(a_n)_{n=1}^{\infty}$ is bounded; by Lemma 28 (or Proposition 27(c)) this implies that the liminf $L_- := \liminf_{n\to\infty} a_n$ and limsup $L_+ := \limsup_{n\to\infty} a_n$ of the sequence are both finite. To show that the sequence converges, it will suffice by Proposition 27(f) to show that $L_- = L_+$.

- Now let $\varepsilon > 0$ be any real number. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, it must be eventually ε-steady, so in particular there exists an $N \geq 1$ such that the sequence $(a_n)_{n=N}^{\infty}$ is ε-steady. In particular, we have $a_N - \varepsilon \leq a_n \leq a_N + \varepsilon$ for all $n \geq N$. By Proposition 24 (or Lemma 28) this implies that

$$a_N - \varepsilon \leq \inf(a_n)_{N=m}^{\infty} \leq \sup(a_n)_{N=m}^{\infty} \leq a_N + \varepsilon$$

and hence by the definition of L_- and L_+ (and Proposition 24 again)

$$a_N - \varepsilon \le L_- \le L_+ \le a_N + \varepsilon.$$

Thus we have

$$0 \le L_+ - L_- \le 2\varepsilon.$$

But this is true for all $\varepsilon > 0$, and L_+ and L_- do not depend on ε; so we must therefore have $L_+ = L_-$. (If $L_+ > L_-$ then we could set $\varepsilon := (L_+ - L_-)/3$ and obtain a contradiction). By Proposition 27(f) we thus see that the sequence converges. □

- In the language of metric spaces (which you will learn about in Math 121), Theorem 30 asserts that the real numbers are a *complete* metric space - that they do not contain "holes" the same way the rationals do. (Certainly the rationals have lots of Cauchy sequences which do not converge to other rationals; take for instance $3, 3.1, 3.14, 3.141, 3.1415, \ldots$). This property is closely related to the least upper bound property discussed in Week 2 notes, and is one of the principal characteristics which make the real numbers superior to the rational numbers for the purposes of doing analysis (taking limits, taking derivatives and integrals, finding zeroes of functions, that kind of thing), as we shall see in later notes.