Math 131AH - Week 10
Textbook pages: 120-133.
Topics covered:

- The first fundamental theorem of calculus
- The second fundamental theorem of calculus
- Products and absolute values of Riemann integrable functions
- The change of variables formula
- Integration by parts

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A little more on the Riemann-Stieltjes integral

- For sake of completeness, and because one of the lemmas here will be useful later, we now give more detail on the Riemann-Stieltjes integral than what we gave in last week's notes. Briefly, the theory of this integral is almost identical to that of the Riemann integral, except that the notion of length of an interval must be replaced by a more general version of α-length.

- One of the key theorems from last week's notes - Theorem 3, to be precise - concerned length and partitions, and in particular showed that $|I| = \sum_{J \in \mathbf{P}} |J|$ whenever $\mathbf{P}$ was a partition of I. We now generalize this slightly.

- **Definition** Let I be an generalized interval, and let $\alpha : X \to \mathbf{R}$ be a function defined on some domain X which contains I. Then we define the expression $\alpha[I]$ as follows. If I is a point or the empty set, we set $\alpha[I] = 0$. If I is an interval of the form $[a, b]$, $[a, b)$, $(a, b]$, or (a, b), then we set $\alpha[I] = \alpha(b) - \alpha(a)$. We refer to $\alpha[I]$ as the *α-length of I*.

- **Example** Let $\alpha : \mathbf{R} \to \mathbf{R}$ be the function $\alpha(x) := x^2$. Then $\alpha[[2, 3]] = \alpha(3) - \alpha(2) = 9 - 4 = 5$, while $\alpha[(-3, -2)] = -5$. Meanwhile $\alpha[\{2\}] = 0$ and $\alpha[\emptyset] = 0$.

- **Example** Let $\alpha : \mathbf{R} \to \mathbf{R}$ be the identity function $\alpha(x) := x$. Then $\alpha[I] = |I|$ for all generalized intervals I (why?) Thus the notion of length is a special case of the notion of α-length.

- We sometimes write $\alpha|_a^b$ or $\alpha(x)|_{x=a}^{x=b}$ instead of $\alpha[[a,b]]$.

- We now generalize Theorem 3 as follows.

- **Lemma 1.** Let I be a generalized interval, let $\alpha : X \to \mathbf{R}$ be a function defined on some domain X which contains I, and let $\mathbf{P}$ be a partition of I. Then we have
$$\alpha[I] = \sum_{J \in \mathbf{P}} \alpha[J].$$

- **Proof.** This is exactly the same as the proof of Theorem 3 in Week 9 notes, the only difference being that we must replace $|I|$ with $\alpha[I]$, $|K|$ with $\alpha[K]$, etc. The one thing we have to check is that
$$\alpha[I] = \alpha[K] + \alpha[I - K]$$
when I is $[a,b]$, $[a,b)$, $(a,b]$, or (a,b), K is $[c,b]$, $[c,b)$, $(c,b]$, or (c,b), and $a \leq c \leq b$. But this amounts to verifying the identity
$$\alpha(b) - \alpha(a) = (\alpha(b) - \alpha(c)) + (\alpha(c) - \alpha(a))$$
but this follows from the laws of algebra. $\square$

- We can now define a generalization of the piecewise constant integral from last week's notes.

- **Definition.** Let I be a generalized interval, and let $\mathbf{P}$ be a partition of I. Let $\alpha : X \to \mathbf{R}$ be a function defined on some domain X which contains I, and let $f : I \to \mathbf{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Then we define
$$p.c. \int_{[\mathbf{P}]} f \, d\alpha := \sum_{J \in \mathbf{P}} c_J \alpha[J]$$
where c_J is the constant value of f on J.

- Compare this to the definition on page 5 of Week 9 notes.

- **Example.** Let $f : [1,3] \to \mathbf{R}$ be the function

$$f(x) = \begin{cases} 4 & \text{when } x \in [1,2) \\ 2 & \text{when } x \in [2,3], \end{cases}$$

let $\alpha : \mathbf{R} \to \mathbf{R}$ be the function $\alpha(x) := x^2$, and let $\mathbf{P}$ be the partition $\mathbf{P} := \{[1,2), [2,3]\}$. Then

$$p.c. \int_{[\mathbf{P}]} f \; d\alpha = c_{[1,2)}\alpha[[1,2)] + c_{[2,3]}\alpha[[2,3]]$$

$$= 4(\alpha(2) - \alpha(1)) + 2(\alpha(3) - \alpha(2)) = 4 \times 3 + 2 \times 5 = 22.$$

- **Example.** Let $\alpha : \mathbf{R} \to \mathbf{R}$ be the identity function $\alpha(x) := x$. Then for any generalized interval I, any partition $\mathbf{P}$ of I, and any function f piecewise constant with respect to $\mathbf{P}$, we have $p.c. \int_{[\mathbf{P}]} f \; d\alpha = p.c. \int_{[\mathbf{P}]} f$ (why?).

- By repeating the proof of Proposition 7 of Week 9 notes, we can obtain an exact analogue of Proposition 7 in which all the integrals $p.c. \int_{[\mathbf{P}]} f$ are replaced by $p.c. \int_{[\mathbf{P}]} f \; d\alpha$. We can thus define $p.c. \int_I f \; d\alpha$ for any piecewise constant function $f : I \to \mathbf{R}$ and any $\alpha : X \to \mathbf{R}$ defined on a domain containing I, in analogy to before, by the formula

$$p.c. \int_I f \; d\alpha := p.c. \int_{[\mathbf{P}]} f \; d\alpha$$

for any partition $\mathbf{P}$ on I with respect to which f is piecewise constant.

- Up until now we have made no assumption on α. Let us now assume that α is *monotone increasing*, i.e. $\alpha(y) \geq \alpha(x)$ whenever $x, y \in X$ are such that $y \geq x$. This implies that $\alpha(I) \geq 0$ for all generalized intervals in X (why?). From this one can easily verify that all the results from Theorem 8 from last week's notes continue to hold when the integrals $p.c. \int_I f$ are replaced by $p.c. \int_I f \; d\alpha$, and lengths $|I|$ are replaced by α-lengths $\alpha(I)$.

- We can then define upper and lower Riemann-Stieltjes integrals $\overline{\int}_I f\ d\alpha$ and $\underline{\int}_I f\ d\alpha$ whenever $f : I \to \mathbf{R}$ is bounded and α is defined on a domain containing I, by the usual formulae

  $$\overline{\int}_I f\ d\alpha = \inf\{p.c. \int_I g\ d\alpha : g \text{ is a piecewise constant function on } I \text{ which majorizes } f\}$$

  and

  $$\underline{\int}_I f\ d\alpha = \sup\{p.c. \int_I g\ d\alpha : g \text{ is a piecewise constant function on } I \text{ which minorizes } f\}.$$

  We then say that f is *Riemann-Stieltjes integrable on I with respect to α* if the upper and lower Riemann-Stieltjes integrals match, in which case we set

  $$\int_I f\ d\alpha := \overline{\int}_I f\ d\alpha = \underline{\int}_I f\ d\alpha.$$

- As before, when α is the identity function $\alpha(x) := x$ then the Riemann-Stieltjes integral is identical to the Riemann integral; thus the Riemann-Stieltjes integral is a generalization of the Riemann integral. (We shall see another comparison between the two integrals a little later, in Corollary 7). Because of this, we sometimes write $\int_I f$ as $\int_I f\ dx$ or $\int_I f(x)\ dx$.

- Most (but not all) of the remaining theory from Week 9 notes then can be carried over without difficulty, replacing Riemann integrals with Riemann-Stieltjes integrals and lengths with α-lengths. (There are a couple results which break down; Theorem 13(g), Proposition 16, and Proposition 17 are not necessarily true when α is discontinuous at key places (e.g. if f and α are both discontinuous at the same point, then $\int_I f\ d\alpha$ is unlikely to be defined). Also, Theorem 14 is still true, but one has to be careful with the proof; the problem here is that some of the references to the length of $|J_k|$ should remain unchanged, and other references to the length of $|J_k|$ should be changed to the α-length $\alpha(J_k)$ - basically, all of the occurrences of $|J_k|$ which appear inside a summation should be replaced with $\alpha(J_k)$, but the rest should be unchanged). Since we will not use Riemann-Stieltjes integrals that much in this course, we will not go into detail into the subtleties here, but further information can be found in the textbook.

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The two fundamental theorems of calculus

- We now have enough machinery to connect integration and differentiation via the familiar fundamental theorem of calculus. Actually, there are two such theorems, one involving the derivative of the integral, and the other involving the integral of the derivative.

- **First Fundamental Theorem of Calculus.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a Riemann integrable function. Let $F : [a, b] \to \mathbf{R}$ be the function

$$F(x) := \int_{[a,x]} f.$$

  Then F is continuous. Furthermore, if $x_0 \in [a, b]$ and f is continuous at x_0, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

- **Proof.** Since f is Riemann integrable, it is bounded (by definition of Riemann integrability). Thus we have some real number M such that $-M \leq f(x) \leq M$ for all $x \in [a, b]$.

- Now let $x < y$ be two elements of $[a, b]$. Then notice that

$$F(y) - F(x) = \int_{[a,y]} f - \int_{[a,x]} f = \int_{[x,y]} f$$

  by Theorem 13(h) of last week's notes. By Theorem 13(e) of last week's notes we thus have

$$\int_{[x,y]} f \leq \int_{[x,y]} M = p.c. \int_{[x,y]} M = M(y - x)$$

  and

$$\int_{[x,y]} f \geq \int_{[x,y]} -M = p.c. \int_{[x,y]} -M = -M(y - x)$$

  and thus

$$|F(y) - F(x)| \leq M(y - x).$$

This is for $y > x$. By interchanging x and y we thus see that

$$|F(y) - F(x)| \leq M(x - y)$$

when $x > y$. Also, we have $F(y) - F(x) = 0$ when $x = y$. Thus in all three cases we have

$$|F(y) - F(x)| \leq M|x - y|.$$

Now let $x \in [a, b]$, and let $(x_n)_{n=0}^{\infty}$ be any sequence in $[a, b]$ converging to x. Then we have

$$-M|x_n - x| \leq F(x_n) - F(x) \leq M|x_n - x|$$

for each n. But $-M|x_n - x|$ and $M|x_n - x|$ both converge to 0 as $n \to \infty$¡ so by the squeeze test $F(x_n) - F(x)$ converges to 0 as $n \to \infty$, and thus $\lim_{n\to\infty} F(x_n) = F(x)$. Since this is true for all sequences $x_n \in [a, b]$ converging to x, we thus see that F is continuous at x. Since x was an arbitrary element of $[a, b]$, we thus see that F is continuous.

- Now suppose that $x_0 \in [a, b]$, and f is continuous at x_0. Choose any $\varepsilon > 0$. Then by continuity, we can find a $\delta > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for all x in the generalized interval $I := [x_0 - \delta, x_0 + \delta] \cap [a, b]$, or in other words

  $$f(x_0) - \varepsilon \leq f(x) \leq f(x_0) + \varepsilon \text{ for all } x \in I.$$

  We now show that

  $$|F(y) - F(x_0) - f(x_0)(y - x_0)| \leq \varepsilon|y - x_0|$$

  for all $y \in I$, since Proposition 12 from Weeks 7/8 notes will then imply that F is differentiable at x_0 with derivative $F'(x_0) = f(x_0)$ as desired.

  Now fix $y \in I$. There are three cases. If $y = x_0$, then $F(y) - F(x_0) - f(x_0)(y - x_0) = 0$ and so the claim is obvious. If $y > x_0$, then

  $$F(y) - F(x_0) = \int_{[x_0,y]} f.$$

Since $x_0, y \in I$, and I is a connected set, then $[x_0, y]$ is a subset of I, and thus we have

$$f(x_0) - \varepsilon \leq f(x) \leq f(x_0) + \varepsilon \text{ for all } x \in [x_0, y],$$

and thus

$$(f(x_0) - \varepsilon)(y - x_0) \leq \int_{[x_0,y]} f \leq (f(x_0) + \varepsilon)(y - x_0)$$

and so in particular

$$|F(y) - F(x_0) - f(x_0)(y - x_0)| \leq \varepsilon |y - x_0|$$

as desired. The case $y < x_0$ is similar and is left to the reader. $\square$

- **Example.** Recall in HW7 that we constructed a monotone function $f : \mathbf{R} \to \mathbf{R}$ which was discontinuous at every rational and continuous everywhere else. This monotone function is Riemann integrable on $[0, 1]$. If we define $F : [0, 1] \to \mathbf{R}$ by $F(x) := \int_{[0,x]} f$, then F is a continuous function which is differentiable at every irrational number. (It turns out to be non-differentiable at every rational number; this can be shown with the aid of the mean value theorem).

- Very informally, the first fundamental theorem of calculus asserts that

$$(\int_{[a,x]} f)'(x) = f(x)$$

given a certain number of assumptions on f. Roughly, this means that the derivative of an integral recovers the original function. Now we show the reverse, that the integral of a derivative recovers the original function.

- **Definition.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be a function. We say that a function $F : I \to \mathbf{R}$ is an *antiderivative* of f if F is differentiable on I and $F'(x) = f(x)$ for all $x \in I$.

- **Second Fundamental Theorem of Calculus.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a Riemann integrable function. If $F : [a, b] \to \mathbf{R}$ is an antiderivative of f, then

$$\int_{[a,b]} f = F(b) - F(a).$$

- **Proof.** We will use Riemann sums. The idea is to show that

$$U(f, \mathbf{P}) \geq F(b) - F(a) \geq L(f, \mathbf{P})$$

for every partition $\mathbf{P}$ of $[a, b]$. The left inequality asserts that $F(b) - F(a)$ is a lower bound for $\{U(f, \mathbf{P}) : \mathbf{P} \text{ is a partition of } [a, b]\}$, while the right inequality asserts that $F(b) - F(a)$ is an upper bound for $\{L(f, \mathbf{P}) : \mathbf{P} \text{ is a partition of } [a, b]\}$. But by Proposition 12 of last week's notes, this means that

$$\overline{\int}_{[a,b]} f \geq F(b) - F(a) \geq \underline{\int}_{[a,b]} f,$$

but since f is assumed to be Riemann integrable, both the upper and lower Riemann integral equal $\int_{[a,b]} f$. The claim follows.

- It remains to show the bound $U(f, \mathbf{P}) \geq F(b) - F(a) \geq L(f, \mathbf{P})$. We shall just show the first bound $U(f, \mathbf{P}) \geq F(b) - F(a)$; the other bound is similar.

- Let $\mathbf{P}$ be a partition of $[a, b]$. From Lemma 1 we have

$$F(b) - F(a) = \sum_{J \in \mathbf{P}} F[J] = \sum_{J \in \mathbf{P} : J \neq \emptyset} F[J],$$

while from definition we have

$$U(f, \mathbf{P}) = \sum_{J \in \mathbf{P} : J \neq \emptyset} \sup_{x \in J} f(x) |J|.$$

Thus it will suffice to show that

$$F[J] \leq \sup_{x \in J} f(x) |J|$$

for all $J \in \mathbf{P}$ (other than the empty set).

- When J is a point then the claim is clear, since both sides are zero. Now suppose that $J = [c, d], (c, d], [c, d)$, or (c, d) for some $c < d$. Then the left-hand side is $F[J] = F(d) - F(c)$. By the mean-value theorem, this is equal to $(d-c)F'(e)$ for some $e \in J$. But since $F'(e) = f(e)$, we thus have

$$F[J] = (d-c)f(e) = f(e)|J| \leq \sup_{x \in J} f(x)|J|$$

as desired. $\square$

- Of course, as you are all aware, one can use the second fundamental theorem of calculus to compute integrals relatively easily, provided that you can find an anti-derivative of the integrand f. Note that the first fundamental theorem of calculus ensures that every *continuous* Riemann integrable function has an anti-derivative. (For discontinuous functions, the situation is a more complicated; we will return to this issue in Math 131B, when we have a better notion of integral. Also, not every function with an anti-derivative is Riemann integrable; as an example, consider the function $F : [-1, 1] \to \mathbf{R}$ defined by $F(x) := x^2 \sin(1/x^3)$ when $x \neq 0$, and $F(0) := 0$. Then F is differentiable everywhere (why?), so F' has an antiderivative, but F' is unbounded (why?), and so is not Riemann integrable.).

- We now pause to mention the infamous "$+C$" ambiguity in anti-derivatives:

- **Lemma 2.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be a function. Let $F : I \to \mathbf{R}$ and $G : I \to \mathbf{R}$ be two antiderivatives of f. Then there exists a real number C such that $F(x) = G(x) + C$ for all $x \in I$.

- **Proof.** The simplest proof is via the mean-value theorem. If I is the empty set then the claim is trivial, so suppose that I is non-empty. Let $H : I \to \mathbf{R}$ be the function $H := F - G$, then H is differentiable and $H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$ for all $x \in I$. Then if we apply the mean-value theorem to H we see that for every $x < x'$ in I there exists a $y \in [x, x']$ such that $H(x') - H(x) = H'(y)(x' - x) = 0$, thus $H(x) = H(x')$ for all $x, x' \in [a, b]$. Thus H is a constant function (why?), and the claim follows by setting C to be the constant value of H on I. (One can also prove this lemma using the second Fundamental theorem of calculus (how?), but one has to be careful since we do not assume f to be to be Riemann integrable.) $\square$

- A little later in these notes we shall give some consequences of the fundamental theorems of calculus.

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Products and absolute values of Riemann integrals

- We have already given two large classes of Riemann integrable functions; piecewise continuous and piecewise monotone functions. For the next few applications, we shall also need some further ways to create Riemann integrable functions.

- **Theorem 3.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ be a Riemann integrable function. Then the functions $\max(f,g) : I \to \mathbf{R}$ and $\min(f,g) : I \to \mathbf{R}$ defined by $\max(f,g)(x) := \max(f(x), g(x))$ and $\min(f,g)(x) := \min(f(x), g(x))$ are also Riemann integrable.

- **Proof.** We shall just prove the claim for $\max(f,g)$, the case of $\min(f,g)$ being similar. First note that since f and g are bounded, then $\max(f,g)$ is also bounded.

  Let $\varepsilon > 0$. Since $\int_I f = \underline{\int}_I f$, there exists a piecewise constant function $\underline{f} : I \to \mathbf{R}$ which minorizes f on I such that

$$\int_I \underline{f} \geq \int_I f - \varepsilon.$$

  Similarly we can find a piecewise constant $\underline{g} : I \to \mathbf{R}$ which minorizes g on I such that

$$\int_I \underline{g} \geq \int_I g - \varepsilon,$$

  and we can find piecewise functions $\overline{f}$, $\overline{g}$ which majorize f, g respectively on I such that

$$\int_I \overline{f} \leq \int_I f + \varepsilon$$

  and

$$\int_I \overline{g} \leq \int_I g + \varepsilon.$$

In particular, if $h : I \to \mathbf{R}$ denotes the function

$$h := (\overline{f} - \underline{f}) + (\overline{g} - \underline{g})$$

we have

$$\int_I h \leq 4\varepsilon.$$

On the other hand, $\max(\underline{f}, \underline{g})$ is a piecewise constant function on I (why?) which minorizes $\max(f, g)$ (why?), while $\max(\overline{f}, \overline{g})$ is similarly a piecewise constant function on I which majorizes $\max(f, g)$. Thus

$$\int_I \max(\underline{f}, \underline{g}) \leq \underline{\int}_I \max(f, g) \leq \overline{\int}_I \max(f, g) \leq \int_I \max(\overline{f}, \overline{g}),$$

and so

$$0 \leq \overline{\int}_I \max(f, g) - \underline{\int}_I \max(f, g) \leq \int_I \max(\overline{f}, \overline{g}) - \max(\underline{f}, \underline{g}).$$

But we have

$$\overline{f}(x) = \underline{f}(x) + (\overline{f} - \underline{f})(x) \leq \underline{f}(x) + h(x)$$

and similarly

$$\overline{g}(x) = \underline{g}(x) + (\overline{g} - \underline{g})(x) \leq \underline{g}(x) + h(x)$$

and thus

$$\max(\overline{f}(x), \overline{g}(x)) \leq \max(\underline{f}(x), \underline{g}(x)) + h(x).$$

Inserting this into the previous inequality, we obtain

$$0 \leq \overline{\int}_I \max(f, g) - \underline{\int}_I \max(f, g) \leq \int_I h \leq 4\varepsilon.$$

To summarize, we have shown that

$$0 \leq \overline{\int}_I \max(f, g) - \underline{\int}_I \max(f, g) \leq 4\varepsilon$$

for every ε. Since $\overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g)$ does not depend on ε, we thus see that

$$\overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g) = 0$$

and hence that $\max(f,g)$ is Riemann integrable. $\square$

- **Corollary 4.** Let I be a generalized interval. If $f : I \to \mathbf{R}$ is a Riemann integrable function, then the positive part $f_+ := \max(f,0)$ and the negative part $f_- := \min(f,0)$ are also Riemann integrable on I. Also, the absolute value $|f| = f_+ - f_-$ is also Riemann integrable on I.

- **Theorem 5.** Let I be a generalized interval. If $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ are Riemann integrable, then $fg : I \to \mathbf{R}$ is also Riemann integrable.

- **Proof.** This one is a little trickier. We split $f = f_+ + f_-$ and $g = g_+ + g_-$ into positive and negative parts; by Corollary 4, the functions f_+, f_-, g_+, g_- are Riemann integrable. Since

$$fg = f_+g_+ + f_+g_- + f_-g_+ + f_-g_-$$

then it suffices to show that the four individual functions $f_+g_+, f_+g_-, f_-g_+, f_-g_-$ are Riemann integrable. We will just show this for f_+g_+; the other three are similar.

- Since f_+ and g_+ are bounded and positive, there are $M_1, M_2 > 0$ such that

$$0 \leq f_+(x) \leq M_1 \text{ and } 0 \leq g_+(x) \leq M_2$$

for all $x \in I$. Now let $\varepsilon > 0$ be arbitrary. Then, as in the proof of Theorem 3, we can find a piecewise constant function $\underline{f_+}$ minorizing f_+ on I, and a piecewise constant function $\overline{f_+}$ majorizing f_+ on I, such that

$$\int_I \overline{f_+} \leq \int_I f_+ + \varepsilon$$

and

$$\int_I \underline{f_+} \geq \int_I f_+ - \varepsilon.$$

Note that $\underline{f_+}$ may be negative at places, but we can fix this by replacing $\underline{f_+}$ by $\max(\underline{f_+}, 0)$, since this still minorizes f_+ (why?) and still has integral greater than or equal to $\int_I f_+ - \varepsilon$ (why?). So without loss of generality we may assume that $\underline{f_+}(x) \geq 0$ for all $x \in I$. Similarly we may assume that $\overline{f_+}(x) \leq M_1$ for all $x \in I$; thus

$$0 \leq \underline{f_+}(x) \leq f_+(x) \leq \overline{f_+}(x) \leq M_1$$

for all $x \in I$.

- Similar reasoning allows us to find piecewise constant $\underline{g_+}$ minorizing g_+, and $\overline{g_+}$ majorizing g_+, such that

$$\int_I \overline{g_+} \leq \int_I g_+ + \varepsilon$$

and

$$\int_I \underline{g_+} \geq \int_I g_+ - \varepsilon,$$

and

$$0 \leq \underline{g_+}(x) \leq g_+(x) \leq \overline{g_+}(x) \leq M_2$$

for all $x \in I$.

Notice that $\underline{f_+ g_+}$ is piecewise constant and minorizes $f_+ g_+$, while $\overline{f_+ g_+}$ is piecewise constant and majorizes $f_+ g_+$. Thus

$$0 \leq \overline{\int}_I f_+ g_+ - \underline{\int}_I f_+ g_+ \leq \int_I \overline{f_+}\,\overline{g_+} - \underline{f_+ g_+}.$$

However, we have

$$\begin{aligned} \overline{f_+}(x)\overline{g_+}(x) - \underline{f_+}(x)\underline{g_+}(x) &= \overline{f_+}(x)(\overline{g_+} - \underline{g_+})(x) + \underline{g_+}(x)(\overline{f_+} - \underline{f_+}(x)) \\ &\leq M_1(\overline{g_+} - \underline{g_+})(x) + M_2(\overline{f_+} - \underline{f_+}(x)) \end{aligned}$$

for all $x \in I$, and thus

$$0 \leq \overline{\int}_I f_+ g_+ - \underline{\int}_I f_+ g_+ \leq M_1 \int_I (\overline{g_+} - \underline{g_+}) + M_2 \int_I (\overline{f_+} - \underline{f_+})$$

$$\leq M_1(2\varepsilon) + M_2(2\varepsilon).$$

Again, since ε was arbitrary, we can conclude that f_+g_+ is Riemann integrable, as before. Similar argument show that f_+g_-, f_-g_+, f_-g_- are Riemann integrable; combining them we obtain that fg is Riemann integrable. $\square$

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Some consequences of the fundamental theorem of calculus

- We can now give a number of useful consequences of the fundamental theorems of calculus. The first is the familiar integration by parts formula.

- **Integration by parts formula.** Let $I = [a,b]$, and let $F : [a,b] \to \mathbf{R}$ and $G : [a,b] \to \mathbf{R}$ be differentiable functions on $[a,b]$ such that F' and G' are Riemann integrable on I. Then we have

$$\int_{[a,b]} FG' = F(b)G(b) - F(a)G(a) - \int_{[a,b]} F'G.$$

- **Proof.** Since F is differentiable on $[a,b]$, it is continuous on the same interval, and hence Riemann integrable (Corollary 15 of last week's notes), as is G. Since F' and G' are Riemann integrable by hypothesis, we see that FG' and $F'G$ are Riemann integrable also by Theorem 5, so both sides make sense.

- Since F and G are differentiable, then so is FG, and we have $(FG)' = F'G + FG'$. From the second fundamental theorem of calculus we thus have

$$\int_{[a,b]} F'G + FG' = FG[[a,b]] = F(b)G(b) - F(a)G(a),$$

  and the claim follows. $\square$

- Next, we show that under certain circumstances, one can write a Riemann-Stieltjes integral as a Riemann integral. We begin with piecewise constant functions.

- **Theorem 6.** Let $\alpha : [a, b] \to \mathbf{R}$ be a monotone increasing function, and suppose that α is also differentiable on $[a, b]$, with α' being Riemann integrable. Let $f : [a, b] \to \mathbf{R}$ be a piecewise constant function on $[a, b]$. Then $f\alpha'$ is Riemann integrable on $[a, b]$, and

$$\int_{[a,b]} f \, d\alpha = \int_{[a,b]} f\alpha'.$$

- **Proof.** Since f is piecewise constant, it is Riemann integrable, and since α' is also Riemann integrable, then $f\alpha'$ is Riemann integrable by Theorem 5.

- Suppose that f is piecewise constant with respect to some partition $\mathbf{P}$ of $[a, b]$; without loss of generality we may assume that $\mathbf{P}$ does not contain the empty set. Then we have

$$\int_{[a,b]} f \, d\alpha = p.c. \int_{[\mathbf{P}]} f \, d\alpha = \sum_{J \in \mathbf{P}} c_J \alpha[J]$$

where c_J is the constant value of f on J. On the other hand, from Theorem 8(h) of Week 9 notes (generalized to partitions of arbitrary length - why is this generalization true?) we have

$$\int_{[a,b]} f\alpha' = \sum_{J \in \mathbf{P}} \int_J f\alpha' = \sum_{J \in \mathbf{P}} \int_J c_J \alpha' = \sum_{J \in \mathbf{P}} c_J \int_J \alpha'.$$

But by the second fundamental theorem of calculus, $\int_J \alpha' = \alpha[J]$, and the claim follows. $\square$

- **Corollary 7.** Let $\alpha : [a, b] \to \mathbf{R}$ be a monotone increasing function, and suppose that α is also differentiable on $[a, b]$, with α' being Riemann integrable. Let $f : [a, b] \to \mathbf{R}$ be a function which is Riemann-Stieltjes integrable with respect to α on $[a, b]$. Then $f\alpha'$ is Riemann integrable on $[a, b]$, and

$$\int_{[a,b]} f \, d\alpha = \int_{[a,b]} f\alpha'.$$

- **Proof.** Note that since f and α' are bounded, then $f\alpha'$ must also be bounded. Also, since α is monotone increasing and differentable, α' is non-negative.

- Let $\varepsilon > 0$. Then, we can find a piecewise constant function $\overline{f}$ majorizing f on $[a,b]$, and a piecewise constant function $\underline{f}$ minorizing f on $[a,b]$, such that

$$\int_{[a,b]} f\ d\alpha - \varepsilon \leq \int_{[a,b]} \underline{f}\ d\alpha \leq \int_{[a,b]} \overline{f}\ d\alpha \leq \int_{[a,b]} f\ d\alpha + \varepsilon.$$

  Applying Theorem 6, we obtain

$$\int_{[a,b]} f\ d\alpha - \varepsilon \leq \int_{[a,b]} \underline{f}\alpha' \leq \int_{[a,b]} \overline{f}\ \alpha' \leq \int_{[a,b]} f\ d\alpha + \varepsilon.$$

- Since α' is non-negative and $\underline{f}$ minorizes f, then $\underline{f}\alpha'$ minorizes $f\alpha'$. Thus $\underline{\int}_{[a,b]}\underline{f}\alpha' \leq \underline{\int}_{[a,b]} f\alpha'$ (why?). Thus

$$\int_{[a,b]} f\ d\alpha - \varepsilon \leq \underline{\int}_{[a,b]} f\alpha'.$$

  Similarly we have

$$\overline{\int}_{[a,b]} f\alpha' \leq \int_{[a,b]} f\ d\alpha + \varepsilon.$$

  Since these statements are true for any $\varepsilon > 0$, we must have

$$\int_{[a,b]} f\ d\alpha \leq \underline{\int}_{[a,b]} f\alpha' \leq \overline{\int}_{[a,b]} f\alpha' \leq \int_{[a,b]} f\ d\alpha$$

  and the claim follows. □

- Informally, Corollary 7 asserts that $f\ d\alpha$ is essentially equivalent to $f \frac{d\alpha}{dx} dx$, when α is differentiable. However, the advantage of the Riemann-Stieltjes integral is that it still makes sense even when α is not differentiable.

- We now build up to the familiar change of variables formula. We first need a preliminary lemma.

- **Lemma 8.** Let $[a,b]$ be a closed interval, and let $\phi : [a,b] \to [\phi(a), \phi(b)]$ be a continuous monotone increasing function. Let $f : [\phi(a), \phi(b)] \to \mathbf{R}$ be a piecewise constant function on $[\phi(a), \phi(b)]$. Then $f \circ \phi : [a,b] \to \mathbf{R}$ is also piecewise constant on $[a,b]$, and

$$\int_{[a,b]} f \circ \phi \; d\phi = \int_{[\phi(a), \phi(b)]} f.$$

- **Proof.** Let $\mathbf{P}$ be a partition of $[\phi(a), \phi(b)]$ such that f is piecewise constant with respect to $\mathbf{P}$; we may assume that $\mathbf{P}$ does not contain the empty set. For each $J \in \mathbf{P}$, let c_J be the constant value of f on J, thus

$$\int_{[\phi(a), \phi(b)]} f = \sum_{J \in \mathbf{P}} c_J |J|.$$

For each generalized interval J, let $\phi^{-1}(J)$ be the set $\phi^{-1}(J) := \{x \in [a,b] : \phi(x) \in J\}$. Then $\phi^{-1}(J)$ is connected (why?), and is thus a generalized interval. Furthermore, c_J is the constant value of $f \circ \phi$ on $\phi^{-1}(J)$ (why?). Thus, if we define $\mathbf{Q} := \{\phi^{-1}(J) : J \in \mathbf{P}\}$ (ignoring the fact that $\mathbf{Q}$ has been used to represent the rational numbers), then $\mathbf{Q}$ partitions $[a,b]$ (why?), and $f \circ \phi$ is piecewise constant with respect to $\mathbf{Q}$ (why?). Thus

$$\int_{[a,b]} f \circ \phi \; d\phi = \int_{[\mathbf{Q}]} f \circ \phi \; d\phi = \sum_{J \in \mathbf{P}} c_J \phi[\phi^{-1}(J)].$$

But $\phi[\phi^{-1}(J)] = |J|$ (why?), and the claim follows. □

- **Change of variables formula, first version.** Let $[a,b]$ be a closed interval, and let $\phi : [a,b] \to [\phi(a), \phi(b)]$ be a continuous monotone increasing function. Let $f : [\phi(a), \phi(b)] \to \mathbf{R}$ be a Riemann integrable function on $[\phi(a), \phi(b)]$. Then $f \circ \phi : [a,b] \to \mathbf{R}$ is Riemann-Stieltjes integrable with respect to ϕ on $[a,b]$, and

$$\int_{[a,b]} f \circ \phi \; d\phi = \int_{[\phi(a), \phi(b)]} f.$$

- **Proof.** This will be obtained from Lemma 8 similarly to how Corollary 7 was obtained from Theorem 6. First observe that since f is Riemann integrable, it is bounded, and then $f \circ \phi$ must also be bounded (why?).

- Let $\varepsilon > 0$. Then, we can find a piecewise constant function $\overline{f}$ majorizing f on $[\phi(a), \phi(b)]$, and a piecewise constant function $\underline{f}$ minorizing f on $[\phi(a), \phi(b)]$, such that
$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \int_{[\phi(a),\phi(b)]} \underline{f} \leq \int_{[\phi(a),\phi(b)]} \overline{f} \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$
Applying Lemma 8, we obtain
$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \int_{[a,b]} \underline{f} \circ \phi \; d\phi \leq \int_{[a,b]} \overline{f} \circ \phi \; d\phi \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$
Since $\underline{f} \circ \phi$ is piecewise constant and minorizes $f \circ \phi$, we have
$$\int_{[a,b]} \underline{f} \circ \phi \; d\phi \leq \underline{\int}_{[a,b]} f \circ \phi \; d\phi$$
while similarly we have
$$\int_{[a,b]} \overline{f} \circ \phi \; d\phi \geq \overline{\int}_{[a,b]} f \circ \phi \; d\phi.$$
Thus
$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \underline{\int}_{[a,b]} f \circ \phi \; d\phi \leq \overline{\int}_{[a,b]} f \circ \phi \; d\phi \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$
Since $\varepsilon > 0$ was arbitrary, this implies that
$$\int_{[\phi(a),\phi(b)]} f \leq \underline{\int}_{[a,b]} f \circ \phi \; d\phi \leq \overline{\int}_{[a,b]} f \circ \phi \; d\phi \leq \int_{[\phi(a),\phi(b)]} f$$
and the claim follows. $\square$

- Combining this formula with Corollary 7, one immediately obtains the following familiar formula:

- **Change of variables formula, second version.** Let $[a, b]$ be a closed interval, and let $\phi : [a, b] \to [\phi(a), \phi(b)]$ be a differentiable monotone increasing function such that ϕ' is Riemann integrable. Let $f : [\phi(a), \phi(b)] \to \mathbf{R}$ be a Riemann integrable function on $[\phi(a), \phi(b)]$. Then $(f \circ \phi)\phi' : [a, b] \to \mathbf{R}$ is Riemann integrable on $[a, b]$, and
$$\int_{[a,b]} (f \circ \phi)\phi' = \int_{[\phi(a),\phi(b)]} f.$$