**Problem 1.** Let $x$ be a real number, and let $n, m$ be natural numbers. Without using any of the exponent laws (other than the definition of exponentiation; see reference sheet), show that $x^{n+m} = x^n x^m$.

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This question is somewhat similar to Q4 of HW1.

We fix $n$ and prove by induction on $m$. When $m = 0$, we have to show that $x^{n+0} = x^n x^0$. But the left-hand side is $x^{n+0} = x^n$, while the right-hand side is $x^n \times 1 = x^n$, so we are done when $m = 0$.

Now suppose inductively that we have already proven that $x^{n+m} = x^n x^m$; we now wish to show $x^{n+(m++)} = x^n x^{m++}$. The left-hand side is $x^{(n+m)++} = x^{n+m} \times x$ by definition of exponentiation; by induction hypothesis this is equal to $(x^n x^m)x$. Meanwhile, the right-hand side is $x^n x^{m++} = x^n(x^m x)$. The two sides are thus equal thanks to the associativity of multiplication.

(One can also proceed by inducting on $n$ instead of $m$, but one then also needs to use the fact that multiplication is commutative as well as associative).

**Problem 2.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, such that $a_{n++} > a_n$ for each natural number $n$. Prove that whenever $n$ and $m$ are natural numbers such that $n > m$, then we have $a_n > a_m$.

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This question is somewhat similar to Q4(a) of HW2.

For this problem it makes a difference which variable to induct on. Inducting on $m$ will lead to trouble; but inducting on $n$ is not too bad. Here's how that goes. Fix $m$, and let $P(n)$ be the property that "if $n > m$, then $a_n > a_m$". The property $P(0)$ is vacuously true (because $0 > m$ is false). Now suppose inductively that $P(n)$ is true; we need to show that $P(m++)$ is true. There are three cases: $n++ \leq m$, $n++ = m++$, and $n++ > m++$. If $n++ \leq m$ then $P(m++)$ is again vacuously true. If $n++ = m++$ then we need to show that $a_{m++} > a_m$, but this is true by hypothesis. If $n++ > m++$, then we also have $n > m$, and so by the inductive hypothesis $P(n)$ we have $a_n > a_m$. But by hypothesis we also have $a_{n++} > a_n$, thus by transitivity of order $a_{n++} > a_m$, and hence $P(n++)$ is also true. Thus by induction $P(n)$ is true for all $n$, and we are done.

If you are not happy with vacuously true statements, another way to proceed is as follows. Let $n > m$; then $n = m + b$ for some non-zero natural number $b$, and thus $n = m + c++$ for some natural number $c$. To prove that $a_n > a_m$, it thus suffices to show that $a_{m+c++} > a_m$ for all natural numbers $m$ and $c$.

Fix $m$; we induct on $c$. When $c = 0$ we need to show that $a_{m+0++} > a_m$, but this follows from hypothesis since $a_{m++} > a_m$. Now suppose inductively that $a_{m+c++} > a_m$; we need to show that $a_{m+(c++)++} > a_m$. But by hypothesis we have $a_{m+(c++)++} > a_{m+c++}$, so by transitivity of order we have $a_{m+(c++)++} > a_m$ as desired.

A more exotic way to prove this statement is by the well-ordering principle and contradiction. Fix $m$, and let $X$ be the set of all $n$ such that $n > m$ and that $a_n \leq a_m$. If we can show that $X$ is empty, then we are done (why?). Suppose for contradiction that $X$ is not empty. Then by the well-ordering principle, $X$ has a minimum element $n_0$; thus $n_0 > m$ and $a_{n_0} \leq a_m$. Since $a_{m++} > a_m$, we have $n_0 \neq m++$, thus $n_0 = n++$ for some $n > m$. But we have $a_{n_0} = a_{n++} > a_n$, thus by transitivity of order $a_n \leq a_m$. But then this implies that $n \in X$, which contradicts the fact that $n_0$ is the minimum of $X$, since $n < n_0$.

It is also possible (though not so easy) to prove this problem using the principle of infinite descent and contradiction; I'll leave that to you as a challenge.

**Problem 3.** Let $A$ and $B$ be finite sets. Show that $A \cup B$ and $A \cap B$ are also finite sets, and

$$\#(A) + \#(B) = \#(A \cup B) + \#(A \cap B)$$

where $\#(A)$ denotes the number of elements in $A$, etc.

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This question is somewhat similar to Q1 of HW3.

One way to prove this is to use Proposition 1 from week 3/4 notes. Since $A$ and $B$ are finite, then $A \cup B$ is finite from Proposition 1(b); since $A \cap B$ is a subset of $A \cup B$, this implies that $A \cap B$ is finite by Proposition 1(c).

In what follows it may be clearer to understand what is going on by drawing a Venn diagram (this would of course not be part of the formal proof, but definitely aids in visualization).

Note that $A \cup B$ is the union of $B$ and $A \backslash B$ (why?), and that $A \backslash B$ is a subset of $A$ and hence finite by Proposition 1(c). Also, $B$ and $A \backslash B$ are disjoint. Thus by Proposition 1(c) we have

$$\#(A \cup B) = \#(B) + \#(A \backslash B).$$

Adding $\#(A \cap B)$ to both sides we have

$$\#(A \cup B) + \#(A \cap B) = \#(B) + \#(A \backslash B) + \#(A \cap B).$$

But the sets $(A \backslash B)$ and $(A \cap B)$ are disjoint, and their union is $A$ (why?), so by Proposition 1(c) again we have

$$\#(A) = \#(A \backslash B) + \#(A \cap B).$$

Combining these two equations we get the result.

A different way to proceed is by induction. Suppose $A$ has $n$ elements and $B$ has $m$ elements. We need to show that $A \cup B$ and $A \cap B$ are finite, and that $\#(A \cup B) + \#(A \cap B)$ equals $n + m$.

We fix $n$ and induct on $m$. First suppose that $m = 0$. Then $B$ is empty, and so $A \cup B = A$ and $A \cap B = \emptyset$. Thus $\#(A \cup B) + \#(A \cap B) = \#(A) + \#(\emptyset) = n + 0$ as desired.

Now suppose inductively that we have already proven the claim for $m$, and now want to prove it for $m{+}{+}$. Thus, let $B$ be a set with $m{+}{+}$ elements. Let $x$ be any element of $B$; then by Lemma 31 of Week 2 notes $B - \{x\}$ has $m$ elements. Write $B' := B - \{x\}$; if we apply the induction hypothesis to $B'$, we see that $A \cup B'$ and $A \cap B'$ are finite, and

$$\#(A \cup B') + \#(A \cap B') = n + m.$$

Now we have to somehow put $x$ back in. We know that $B = B' \cup \{x\}$. (Again, a Venn diagram may be helpful to understand the proof at this point). By definition of $B'$ we know that $x$ is not an element of $B'$. There are two cases: either $x$ is an element of $A$, or $x$ is not an element of $A$.

If $x$ is an element of $A$, then $A \cup B$ is the same set as $A \cup B'$ (why?), while $A \cap B$ is equal to $A \cap B'$ union $\{x\}$ (why?). By Proposition 1(a) of Week 3/4 notes we thus have $\#(A \cap B) = \#(A \cap B') + 1$, while $\#(A \cup B) = \#(A \cup B')$. Substituting these into the previous equation we obtain $\#(A \cup B) + \#(A \cap B) = n + m++$ as desired.

Now suppose $x$ is not an element of $A$. Then $A \cup B$ is the equal to set $A \cup B'$ union $\{x\}$ (why?), while $A \cap B$ is the same set as $A \cap B'$ (why?). By Proposition 1(a) of Week 3/4 notes we thus have $\#(A \cup B) = \#(A \cup B') + 1$, while $\#(A \cap B) = \#(A \cap B')$. Substituting these into the previous equation we obtain $\#(A \cup B) + \#(A \cap B) = n + m++$ as desired.

**Problem 4.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=0}^{\infty}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=0}^{\infty}$. Show that $(b_n)_{n=0}^{\infty}$ is also bounded.

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This question is somewhat similar to Q5 of HW2.

Note that we do not assume in this problem that $(a_n)_{n=0}^{\infty}$ or $(b_n)_{n=0}^{\infty}$ are Cauchy sequences.

Since $(a_n)_{n=0}^{\infty}$ is bounded, we know that there is a real number $M$ such that $|a_n| \leq M$ for all natural numbers $n$. (Note: when we say a sequence is bounded, we mean that it has *both* an upper bound *and* a lower bound, not just one of the two). Also, since $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent, we know that they are eventually 1-close (for instance), which means that there is a natural number $N$ such that $|a_n - b_n| \leq 1$ for all $n \geq N$.

If $n \geq N$, then from the triangle inequality we have

$$|b_n| \leq |a_n| + |b_n - a_n| = |a_n| + |a_n - b_n| \leq M + 1;$$

note that other permutations of the triangle inequality may not work properly because the inequality signs may go the wrong way. To handle the case $n < N$, we observe that the sequence $(b_n)_{n=0}^{N-1}$ is finite, hence bounded by some number $M'$, i.e. $|b_n| \leq M'$ for all $n < N$. Thus if we set $M'' := \max(M+1, M')$, then we have $|b_n| \leq M''$ for all natural numbers $n$ (in both cases $n \geq N$ and $n < N$). Thus the sequence $(b_n)_{n=0}^{\infty}$ is bounded.

Notice that one has to deal with the case $n < N$ separately, because the two sequences $a_n$ and $b_n$ are only *eventually* 1-close; they aren't necessarily 1-close to start with.

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**Problem 5.** Let $E$ be a subset of the real numbers $\mathbf{R}$, and suppose that $E$ has a least upper bound $M$ which is a real number, i.e. $M = \sup(E)$. Let $-E$ be the set

$$-E := \{-x : x \in E\}.$$

Show that $-M$ is the greatest lower bound of $-E$, i.e. $-M = \inf(-E)$.

This question is not similar to any problem in HW1-3 (though it is similar to Q8(a) on HW4).

We need to show that $-M$ is the greatest lower bound of $-E$. This requires us to do two things. First, we must show that $-M$ is a lower bound for $-E$. Secondly, for any other lower bound $M'$ of $-E$, we must show that $-M \geq M'$.

We show the former first. Let $-x$ be any element of $-E$; we have to show that $-M \leq -x$. But we know that $x \leq M$ since $x \in E$ (by definition of $-E$) and $M$ is an upper bound for $E$. Thus $-M \leq -x$ as desired. (If $x \leq M$ then $M - x$ is positive or zero, hence $-M - (-x)$ is negative or zero, hence $-M \leq -x$).

Now we show the latter. Let $M'$ be another bound of $-E$, thus $M' \leq -x$ for all $x \in E$. Thus $-M' \geq x$ for all $x \in E$, i.e. $-M'$ is an upper bound of $E$. Since $M$ is the least upper bound of $E$, we thus have $-M' \geq M$, and thus $M' \leq -M$, as desired.