Mathematics 131AH
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Instructions: Do nine out of the 12 problems; they are all of equal value. There is plenty of working space, and a blank page at the end. On the next three pages you will be supplied ist of standard definitions for easy reference.

You may use any result from the textbook or notes (or from any other mathematics book); you do not need to give precise theorem numbers or page numbers (e.g. saying "by a theorem from the notes" will suffice).

You may enter in a nickname if you want your final score posted.

Good luck!

Name: ____________________

Nickame: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (10 points). __________

Problem 2 (10 points). __________

Problem 3 (10 points). __________

Problem 4 (10 points). __________

Problem 5 (10 points). __________

Problem 6 (10 points). __________

Problem 7 (10 points). __________

Problem 8 (10 points). __________

Problem 9 (10 points). __________

Problem 10 (10 points). __________

Problem 11 (10 points). __________

Problem 12 (10 points). __________

Best 9 of 12 (90 points): ____________________

Definitions

- **Absolutely convergent series.** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. We say that this series is *absolutely convergent* if the series $\sum_{n=m}^{\infty} |a_n|$ is convergent.

- **Adherent points.** Let X be a subset of $\mathbf{R}$, let $\varepsilon > 0$, and let $x \in \mathbf{R}$. We say that x is *ε-adherent to X* iff there exists a $y \in E$ which is ε-close to x. We say that x is *adherent to X* iff it is ε-adherent to X for every $\varepsilon > 0$.

- **Bounded functions.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *bounded* if there exists a number M such that $|f(x)| \leq M$ for all $x \in X$.

- **Bounded sequences.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. We say this sequence is *bounded* if there exists a number M such that $|a_n| \leq M$ for all $n \in \mathbf{N}$.

- **Closure.** Let X be a subset of $\mathbf{R}$. The *closure* of X, sometimes denoted $\overline{X}$ is defined to be the set of all the adherent points of $\mathbf{R}$.

- **Connected sets.** Let X be a subset of $\mathbf{R}$. We say that X is *connected* iff the following property is true: whenever x, y are elements in X such that $x < y$, the interval $[x, y]$ is a subset of X (i.e. every number between x and y is also in X).

- **Continuity.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. Let x_0 be an element of X. We say that f is *continuous at x_0* iff we have

  $$\lim_{x \to x_0; x \in X} f(x) = f(x_0).$$

  We say that f is *continuous on X* iff f is continuous at x_0 for every $x_0 \in X$.

- **Convergent series.** Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, we define the *N^{th} partial sum S_N* of this series to be $S_N := \sum_{n=m}^{N} a_n$. If the sequence $(S_N)_{n=m}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *convergent*, and *converges to L*; we also write $L = \sum_{n=m}^{\infty} a_n$.

- **Differentiable functions.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. Let $x_0 \in X$ be an element of X which is also a limit point of X. If the limit

  $$\lim_{x \to x_0; x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

  converges to some real number L, then we say that f is *differentiable at x_0 on X with derivative L*, and write $f'(x_0) := L$. If f is differentiable at every element of X, we say that f is *differentiable on X*.

- **Generalized interval.** A *generalized interval* is a subset I of $\mathbf{R}$ which is either an interval (i.e. a set of the form $[a, b]$, (a, b), $[a, b)$, or $(a, b]$); a point $\{a\}$; or the empty set $\emptyset$.

- **Length.** If I is a generalized interval, we define the *length* of I, denoted $|I|$ as follows. If I is one of the intervals $[a, b]$, (a, b), $[a, b)$, or $(a, b]$ for some real numbers $a < b$, then we define $|I| := b - a$. Otherwise, if I is a point or the empty set, we define $|I| = 0$.

- **α-Length.** If I is a generalized interval and α is a monotone increasing function on a domain containing I, we define the *α-length* of I, denoted $\alpha[I]$ as follows. If I is one of the intervals $[a, b]$, (a, b), $[a, b)$, or $(a, b]$ for some real numbers $a < b$, then we define $\alpha[I] := \alpha(b) - \alpha(a)$. Otherwise, if I is a point or the empty set, we define $|I| = 0$.

- **Limiting values of functions.** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let E be a subset of X, x_0 be an adherent point of E, and let L be a number. We say that f *converges to L at x_0 in E*, and write $\lim_{x \to x_0; x \in E} f(x) = L$, iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| \leq \varepsilon$ for all $x \in E$ such that $|x - x_0| < \delta$.

- **Majorizing/Minorizing.** Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. We say that f *majorizes* g if $f(x) \geq g(x)$ for all $x \in X$, and f *minorizes* g if $f(x) \leq g(x)$ for all $x \in X$.

- **Monotone increasing.** Let $f : X \to \mathbf{R}$ be a function. We say that f is *monotone increasing* iff we have $f(y) \geq f(x)$ whenever $x, y \in X$ are such that $y > x$.

- **Partitions.** Let I be a generalized interval. A *partition* of I is a finite set $\mathbf{P}$ of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals J in $\mathbf{P}$.

- **Piecewise constant functions.** Let I be a generalized interval, let $f : I \to \mathbf{R}$ be a function, and let $\mathbf{P}$ be a partition of I. We say that f is *piecewise constant with respect to* $\mathbf{P}$ if for every $J \in \mathbf{P}$, f is constant on J. We say that f is *piecewise constant on I* if it is piecewise constant with respect to some partition $\mathbf{P}$ of I.

- **Piecewise constant integrals.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be a function which is piecewise constant with respect to some partition $\mathbf{P}$ of I. Then we define the *piecewise constant integral p.c.* $\int_I f$ of f by the formula

  $$p.c. \int_I f := \sum_{J \in \mathbf{P} : J \neq \emptyset} c_J |J|,$$

  where for each J we let c_J be the constant value of f on J. More generally, if α is a monotone increasing function on a domain containing I, we define

  $$p.c. \int_I f \, d\alpha := \sum_{J \in \mathbf{P} : J \neq \emptyset} c_J \alpha[J].$$

- **Riemann integral.** Let $f : I \to \mathbf{R}$ be a bounded function defined on a generalized interval I. We define the *upper Riemann integral* $\overline{\int}_I f$ by the formula

  $$\overline{\int}_I f := \sup\{p.c. \int_I g : g \text{ is a piecewise constant function on } I \text{ which majorizes } f\}$$

and the *lower Riemann integral* $\underline{\int}_I f$ by the formula

$$\underline{\int}_I f := \sup\{p.c. \int_I g : g \text{ is a piecewise constant function on } I \text{ which minorizes } f\}.$$

If $\overline{\int}_I f = \underline{\int}_I f$, we say that f is *Riemann integrable* and write $\int_I f = \overline{\int}_I f = \underline{\int}_I f$.

- **Riemann-Stieltjes integral.** Let $f : I \to \mathbf{R}$ be a bounded function defined on a generalized interval I. We define the *upper Riemann-Stieltjes integral* $\overline{\int}_I f \; d\alpha$ by the formula

$$\overline{\int}_I f \; d\alpha := \sup\{p.c. \int_I g \; d\alpha : g \text{ is a piecewise constant function on } I \text{ which majorizes } f\}$$

and the *lower Riemann-Stieltjes integral* $\underline{\int}_I f \; d\alpha$ by the formula

$$\underline{\int}_I f \; d\alpha := \sup\{p.c. \int_I g \; d\alpha : \text{ is a piecewise constant function on } I \text{ which minorizes } f\}.$$

If $\overline{\int}_I f \; d\alpha = \underline{\int}_I f \; d\alpha$, we say that f is *Riemann-Stieltjes integrable* and write $\int_I f \; d\alpha = \overline{\int}_I f \; d\alpha = \underline{\int}_I f \; d\alpha$.

- **Uniformly continuous functions.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *uniformly continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x, x_0 \in X$ are two points in x are δ-close.