Assignment 8 (Due Mar 7). Covers: Week 7/8 notes

- Q1. Prove Proposition 12 from Week 7/8 notes. (Note: the cases $x = x_0$ and $x \neq x_0$ have to be treated separately).

- Q2. Prove Proposition 13 from Week 7/8 notes. (Hint: Use the limit laws. Alternatively, one can argue using Proposition 12).

- Q3 (a). Prove Theorem 15 from Week 7/8 notes. (Hint: Use the limit laws. Use earlier parts of this theorem to prove the latter. For the product rule, use the identity

$$f(x)g(x) - f(x_0)g(x_0) = f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)$$

$$= f(x)(g(x) - g(x_0)) + (f(x) - f(x_0))g(x_0);$$

this trick of adding and subtracting an intermediate term is sometimes known as the "middle-man trick" and is very useful in analysis).

- Q3 (b). Let $n$ be a natural number, and let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) := x^n$. Show that $f$ is differentiable on $\mathbf{R}$ and $f'(x) = nx^{n-1}$ for all $x \in \mathbf{R}$. (Hint: Use Theorem 15 and induction).

- Q3 (c). Let $n$ be a negative integer, and let $f : \mathbf{R} - \{0\} \to \mathbf{R}$ be the function $f(x) := x^n$. Show that $f$ is differentiable on $\mathbf{R}$ and $f'(x) = nx^{n-1}$ for all $x \in \mathbf{R} - \{0\}$. (Hint: Use Theorem 15 and Q3(b)).

- Q4*. Prove Theorem 16 from Week 7/8 notes. (Hint: Use Proposition 6 from Week 6 notes to convert this problem into one involving limits of sequences. Note from Proposition 13 that if $(x_n)_{n=1}^\infty$ is a sequence in $X$ which converges to $x_0$, then $(f(x_n))_{n=1}^\infty$ will be a sequence in $Y$ which converges to $f(x_0) = y_0$).

- Q5 (a). Prove Proposition 17 from Week 7/8 notes.

- Q5 (b). Give an example of a function $f : (-1, 1) \to \mathbf{R}$ which is continuous and attains a global maximum at 0, but which is not differentiable at 0. Explain why this does not contradict your answer to Q5(a).

- Q6 (a). Prove Theorem 18 from Week 7/8 notes. (Hint: use Corollary 14 and the Maximum principle, followed by Proposition 17. Note that the maximum principle does not tell you whether the maximum or minimum is in the open interval $(a, b)$ or is one of the boundary points $a, b$, so you have to divide into cases and use the hypothesis $g(a) = g(b)$ somehow).

- Q6 (b). Use Theorem 18 to prove Corollary 19 from Week 7/8 notes. (Hint: Apply Theorem 18 with the function $g : [a, b] \to \mathbf{R}$ defined by $g(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$).

- Q7(a). Prove Proposition 20 from Week 7/8 notes.

- Q7(b). Give an example of a function $f : (-1, 1) \to \mathbf{R}$ which is continuous and monotone increasing, but which is not differentiable at 0. Explain why this does not contradict your answer to Q7(a).

- Q8(a). Prove Proposition 21 from Week 7/8 notes. (Hint: You do not have integrals or the fundamental theorem of calculus yet, so these tools cannot be used. However, one can proceed via the mean-value theorem).

- Q8(b). Give an example of a subset $X \subset \mathbf{R}$ and a function $f : X \to \mathbf{R}$ which is differentiable on $X$, is such that $f'(x) > 0$ for all $x \in X$, but $f$ is not strictly monotone increasing. (Hint: The conditions here are subtly different from those in Proposition 21. What is the difference, and how can one exploit that difference to obtain the example?)

- Q9 (a). Let $n \geq 1$, and let $g : (0, \infty) \to (0, \infty)$ be the function $g(x) := x^{1/n}$. Show that $g$ is continuous on $(0, \infty)$. (Hint: use Proposition 3).

- Q9 (b). Let $g$ be as in (a). Show that $g$ is differentiable on $(0, \infty)$, and that $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$ for all $x \in (0, \infty)$. (Hint: use the inverse function theorem and (a)).

- Q9 (c). Let $q$ be a rational number, and let $f : (0, \infty) \to \mathbf{R}$ be the function $f(x) = x^q$. Show that $f$ is differentiable on $(0, \infty)$ and that $f'(x) = qx^{q-1}$. (Hint: use Q3, Q9(b), and the chain rule.)

- Q9 (d). Show that $\lim_{x \to 1; x \in (0,\infty)} \frac{x^q - 1}{x-1} = q$ for every rational number $q$. (Hint: Use Q9(c) and either the definition of derivative, of L'Hôpital's rule).

- Q9 (e). Show that $\lim_{x \to 1; x \in (0,\infty)} \frac{x^\alpha - 1}{x-1} = \alpha$ for every *real* number $\alpha$. (Hint: Use Q9(d) and the comparison principle; you may need to consider right and left limits separately. Proposition 25 from Week 2 notes may also be helpful).

- Q9 (f). Let $\alpha$ be a real number, and let $f : (0, \infty) \to \mathbf{R}$ be the function $f(x) = x^\alpha$. Show that $f$ is differentiable on $(0, \infty)$ and that $f'(x) = \alpha x^{\alpha - 1}$. (Hint: use Q9(e), exponent laws (Proposition 31 of Week 5 notes), and the definition of derivative).

- Q10. Prove Proposition 24 from Week 7/8 notes. (To show that $g(x) \neq 0$ near $x_0$, you may wish to use Newton's approximation (Proposition 12). For the rest of the Proposition, use limit laws).