Assignment 5 (Due Feb 14). Covers: Week 5 notes

Note: For this assignment you may freely use any material covered in Weeks 1-4, and you can use material from Week 5 as long as it avoids circularity (e.g. using later results to prove earlier results). In particular, for Q10 you may use anything in Week 1-5 notes. Note that the use of logarithms are not allowed, since they have not yet been introduced.

Also, some of the results here are also proven in the textbook (though the notation is a little different sometimes). You are encouraged to read the textbook; you are welcome to use the proofs from that textbook provided you phrase them in your own words.

- Q1. Prove Lemma 2 from Week 5 notes. (Some hints: Review the proof of Proposition 28 from Week 2 notes. Also, you will find proof by contradiction a useful tool, especially when combined with the trichotomy of order and the exponent laws from Week 2 notes. The earlier parts of the Lemma can be used to prove later parts of the Lemma. With part (e), first show that if $x > 1$ then $x^{1/n} > 1$, and if $x < 1$ then $x^{1/n} < 1$).

- Q2. Prove Lemma 4 from Week 5 notes. (Hint: you should rely mainly on Lemma 2 and on algebra).

- Q3 (a). Show that for any $\varepsilon > 0$, the sequence $((1+\varepsilon)^n)_{n=0}^{\infty}$ does not have any real upper bound $M$. (Hint: set $x := 1/(1+\varepsilon)$ and use the fact (from Week 3/4 notes) that $\lim_{n\to\infty} x^n = 0$).

- Q3 (b). Use (a) to show that for every $\varepsilon > 0$ and every real number $M$, there exists an $n$ such that $M^{1/n} \leq 1 + \varepsilon$.

- Q3 (c). Use (b) to prove Lemma 6 of Week 5 notes. (Hint: You may need to treat the cases $x \geq 1$ and $x < 1$ separately).

- Q4. Prove Lemma 8 of Week 5 notes. (Hint: You will need to use induction, but you may need to first think about how to modify the principle of induction to work in situations where the base case is not 0. Alternatively, you may need to make some sort of substitution to make the base case 0 again).

- Q5. Prove Proposition 10 of Week 5 notes. (Hint: This is largely a matter of choosing the right bijections to turn these sums over sets into finite series, and then applying Lemma 8.)

- Q6 (a). Prove Proposition 13 of Week 5 notes. (Hint: use Propositions 18 of Week 3/4 notes and Proposition 7 of Week 5 notes).
- Q6 (b). Use (a) to prove Corollary 14 of Week 5 notes.
- Q6 (c). Use (a) to prove Proposition 15 of Week 5 notes. (Hint: Use Lemma 8(e) of Week 5 notes).
- Q7. Prove Proposition 17 of Week 5 notes.
- Q8. Prove Corollary 19 of Week 5 notes.
- Q9. Prove the first inequality in Lemma 27 of Week 5 notes.
- Q10. Let $x$ be a real number with $|x| < 1$, and $q$ be a real number. Show that the series $\sum_{n=1}^{\infty} n^q x^n$ is absolutely convergent, and that $\lim_{n\to\infty} n^q x^n = 0$.