Assignment 4 (Due Feb 7). Covers: Week 3/4 notes

Note: For this assignment you may freely use any material covered in Weeks 1-3, including all the familiar laws of high-school algebra for all number systems up to and including the reals. For Q9-10, you can use any material from Week 1-4 or from the textbook.

Also, some of the results here are also proven in the textbook (though the notation is a little different sometimes). You are encouraged to read the textbook; you are welcome to use the proofs from that textbook provided you phrase them in your own words.

- Q1. Prove Proposition 18 from the Week 3/4 notes.

- Q2. Prove Proposition 19 from the Week 3/4 notes, using the following outline. Let $(a_n)_{n=m}^{\infty}$ be a Cauchy sequence of rationals, and write $L := LIM_{n\to\infty} a_n$. We have to show that $(a_n)_{n=m}^{\infty}$ converges to L. Let $\varepsilon > 0$. Assume for contradiction that sequence a_n is *not* eventually ε-close to L. Use this, and the fact that $(a_n)_{n=m}^{\infty}$ is Cauchy, to show that there is an $N \geq m$ such that either $a_n > L + \varepsilon/2$ for all $n \geq N$, or $a_n < L - \varepsilon/2$ for all $n \geq N$. Then somehow use Corollary 22 and Proposition 25 from Week 2 notes to get the contradiction.

- Q3. Prove Theorem 21 from the Week 3/4 notes. (Hint: You can use some parts of the theorem to prove others, e.g. (b) can be used to prove (c); (a),(c) can be used to prove (d); and (b), (e) can be used to prove (f). The proofs are similar to those of Lemma 11, Proposition 13, and Lemma 16 from Week 2 notes. For (e), you may need to first prove the auxiliary result that any sequence whose elements are non-zero, and which converges to a non-zero limit, is bounded away from zero).

- Q4. (a) Prove Theorem 23 from the Week 3/4 notes. (You may need to break into cases depending on whether $+\infty$ or $-\infty$ lives in E. You can of course use the fact from Week 2 notes that $\sup(E)$ is the least upper bound for E *provided that E consists only of real numbers*).

- Q4. (b) Using Theorem 23, prove Proposition 24 from the Week 3/4 notes.

- Q5. Prove Proposition 25 from the Week 3/4 notes. (Hint: Set $x := \sup(a_n)_{n=m}^{\infty}$, and use Proposition 24, together with the assumption that a_n is increasing, to show that a_n converges to x).
- Q6. Prove Proposition 26 from the Week 3/4 notes.
- Q7. Prove Proposition 27(cdef) from the Week 3/4 notes. (Hint: you can use earlier parts of the proposition to prove later ones).
- Q8. (a) Prove Lemma 28 from the Week 3/4 notes. (Hint: use the first two inequalities to prove the last two inequalities).
- Q8. (b) Use Lemma 28 to prove Corollary 29.
- Q9. Give an example of two bounded sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $a_n < b_n$ for all $n \geq 1$, but that $\sup(a_n)_{n=1}^{\infty} \not< \sup(b_n)_{n=1}^{\infty}$. Explain why this does not contradict Lemma 28 of the Week 3/4 notes.
- Q10. Do Question 1 on Page 78 of the textbook.