1. Solutions to HW #9

Problem #1

(b) implies (a) is obvious (in the sense that it is a direct checking of definitions), so we concentrate on (a) implies (b). Further, we may assume X is not empty or the set $\{point\}$ as these cases are obvious. As X is bounded, $sup\{x \in X\} = M$ and $inf\{x \in X\} = m$ exist. By the definition of M and m, for all $\epsilon > 0$ we can find $m \leq x \leq m + \epsilon$ and $M \geq y \geq M - \epsilon$ with $x, y \in X$. By connectedness, $[x, y] \subset X$. We conclude that in fact $(m, M) \subseteq X$. On the other hand, if there were $z < m$ or $w > M \in X$ we would have a contradiction. Therefore $X \cap ((M, \infty) \cup (-\infty, m)) = \emptyset$. So X must be a generalized interval.
By the above, if I and J are generalized intervals, they are bounded and connected. Then it is clear that $I \cap J$ is bounded (Why?). Also if $x, y \in I \cap J$ then $[x, y] \subset I$ and $[x, y] \subset J$, so $[x, y] \subset I \cap J$, so $I \cap J$ is connected. But then we may apply the above once again going the other way to conclude that $I \cap J$ is a generalized interval.

Problem # 6

Using the hint, it is enough to show that if $\mathbf{P}'$ refines $\mathbf{P}$, where each is a partition of H, then $p.c.\int_{[\mathbf{P}']} f = p.c.\int_{[\mathbf{P}]} f$. $p.c.\int_{[\mathbf{P}]} f = \sum_{J \in \mathbf{P}} c_J |J|$. Note that $\{I \in P' : I \subset J\}$ partitions each $J \in P$: Fix J. If $x \in J$, then as P' is a partition of H, there is one and only one $I \in P'$ with $x \in I$. By definition of P' being a refinement, $I \subset J$ and so $I \in P'$ with $x \in I$ refines J.
Now Theorem 3 says $|J| = \sum_{I \in \mathbf{P}', I \subset J} |I|$ since the latter set of generalized intervals was just join to refine J. Thus $\sum_{J \in \mathbf{P}} c_J |J| = \sum_{J \in \mathbf{P}} c_J (\sum_{I \in \mathbf{P}', I \subset J} |I|)$. Since $f|_{I \subset J} = c_J$, this last sum is exactly $p.c.\int_{[\mathbf{P}']} f$.

Problem # 4

If P is a partition for f and P' is a partition refining P, let $J \in P'$. Then by definition, there exists $I \in P$ with $J \subset I$. As $f|_I \equiv c_I$, $f|_J \equiv c_I$ also. So f is piecewise constant with respect to P'.