1. Solutions to HW #8

Problem #2

Using Proposition 12, if the function f is differentiable at x_0 then fix $\epsilon > 0$. Then we have the existence of a δ neighborhood of x_0 such that the estimate $|f(x) - f(x_0)| \leq M + \epsilon|x - x_0|$ if x is in the δ neighborhood. In particular we may take $\epsilon < 1$ in the estimate. Thus $|f(x) - f(x_0)| \leq (M+1)\delta$ if x is in the δ neighborhood. Thus we are done (Why?).

Problem # 5(a)

We prove that at any maximum x_0 of the differentiable function f, $f'(x_0) = 0$. We know that the left sided limit and the right sided limit of $\frac{f(x)-f(x_0)}{x-x_0}$ exist and are equal. Assume that we are given δ such that if $|x - x_0| < \delta$ then $f(x) - f(x_0) \leq 0$. In this neighborhood of x_0, $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$ if $x < x_0$ and $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$ if $x > x_0$. But then $\lim_{x \to x_0, x > x_0} \frac{f(x)-f(x_0)}{x-x_0} \leq \sup_{x > x_0, |x-x_0| < \delta} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$ and $\lim_{x \to x_0, x < x_0, |x - x_0 < \delta} \frac{f(x)-f(x_0)}{x-x_0} \leq \inf_{x < x_0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$. So we must have $\lim_{x\ x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0$. The proof is similar in the case that x_0 is a minimum.

Problem # 8(a) All three cases are variants of the same argument, so we present only one. Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) > 0$ whenever $x \in [a,b]$. If $z < w$ are in $[a,b]$, then the MVT applies by previous HW problems (what is the issue here?) . We have $\frac{f(w)-f(z)}{w-z} = f'(c) > 0$ with $c \in [z,w]$. As $w - z > 0$ a little algebra implies $f(w) > f(z)$.