1. Solutions to HW #7

Problem #2

First we show that $f|_Y$ is continuous if $Y \subseteq X$ and f is continuous on X. There are two issues here. One is that f and $f|_Y$ are not the same function, they have different domains. Second is that the definition of continuity for X cannot be applied directly to Y. However there is a nice short proof using things we have already proved. Let i_Y be the injection of Y into X. Then it is easy to show i_Y is continuous. Now $f|_Y = f \circ i_Y$, therefore by the final problem of Week 6 $f|_Y$ is continuous and we are done. Next we use this to show that for any g, continuous on $[a, b]$, if $M = \sup_{x\in[a,b]}\{f(x)\}$ and $m = \inf_{x\in[a,b]}\{f(x)\}$ then for all $m \leq y \leq M$ there exists a $c \in [a, b]$ with $f(c) = y$. First we note that the extreme value theorem gives the existence of e, d such that $f(e) = m, f(d) = M$. Assume for definiteness that $e \leq f$. By the first part, $f|_{[e,f]}$ is continuous, therefore we can apply the intermediate value theorem to this function. Thus for all $m \leq y \leq M$ there exists $c \in [e, f] \subseteq [a, b]$ with $f(c) = y$.

Problem #5 Assume that $f(a) < f(b)$. We first show that for all $c \in (a, b)$, $f(a) < f(c) < f(b)$. For a contradiction, suppose not. Clearly, we may assume either $f(c) < f(a) < f(b)$ or $f(c) > f(a) > f(b)$. The cases are similar so we show a contradiction only in the first. In this case, we apply the above problem to $f_{[c,b]}$ to find an x such that $f(x) = f(a)$. But this is a contradiction since f was assumed to be 1-1 and $a \neq x$. Next, note that the same result can be applied to any subinterval $[x, y]$ of $[a, b]$. Finally, if f were not strictly increasing then there exist $z < w \in [a, b]$ such that $f(a) < f(w) \leq f(z)$, contradicting the preceding sentence. The case $f(a) > f(b)$ is similar and omitted and $f(a) = f(b)$ needs no comment.

Problem #10 Most people got this one so here is a slightly sketchy version. If $f(E)$ were not bounded, then we could find a sequence $\{x_n\}$ such that $|f(x_n| \geq N$. On the other hand E is bounded, so by the Balzano-Weierstrauss Theorem, we can find a subsequence $\{x_{n_j}\}$ that converges (to a point that is not necessarily in E. But then as $\{x_{n_j}\}$ is Cauchy, the previous problem tells us that $\{f(x_{n_j})\}$ is also, and hence is bounded, a contradiction.