1. Solutions to HW6

Problem #4

Suppose x is adherent to X. Then for all $\epsilon > 0$ we can find $a(\epsilon)$ such that $|a(\epsilon) - x| < \epsilon$. Then consider the sequence with terms $b_n = a(\frac{1}{n})$. Now given $\delta > 0$ we can find M such that if $n \geq M$, $|b_n - x| < \frac{1}{n} \leq \delta$. We have therefore found a sequence of elements, lying entirely in X, converging to x. The converse is just an inversion of this proof.

Problem #6

We have to prove

$$\lim_{x \to x_0, x \in X} f(x) = L \Leftrightarrow \lim_{x \to x_0, x \in X \cap |x - x_0| < \delta} f(x) = L$$

. The forward implication is truly obvious, so we worry about "right to left". Suppose that $\{x_n\}$ is a sequence of terms in X, not necessarily δ close to x_0, converging to x_0. Then by definition, there exists M such that the sequence $\{x_n\}_{n \geq M}\}$ is δ close to x_0. Therefore, we may apply the hypothesis and conclude that $\lim_{n \to \infty n \geq M} f(x_n) = L$. But this is the same as $\lim_{n \to \infty} f(x_n) = L$. Thus for any sequence $\{x_n\}$ converging to x_0, $\{f(x_n)\}$ converges to $f(x_0)$. Using various results about the equivalence of convergence to sequential convergence, we are done.

Problem # 10

We will prove that if $f : X \to Y$ is continuous at x_0 and $g : Y \to \mathbb{R}$ at $f(x_0)$, then $g \circ f$ is continuous at x_0. Let $\epsilon > 0$ be given. We must produce δ such that if $|x - x_0| < \delta$ then $|g(f(x)) - g(f(x_0))| < \epsilon$. As g is continuous, there exists δ_1 with $|g(y) - g(f(x_0)| < \epsilon$ if $|y - f(x_0) < \delta_1$. Since f is continuous, we can find δ_2 such that if $|x - x_0| < \delta_2$ then $|f(x) - f(x_0)| < \delta_1$. Then setting $\delta = \delta_2$ does the job.