1. Solutions to HW5

Problem #4 parts d and e

- If $q > 0$ then $x > y$ iff $x^q > y^q$.
  If $q > 0$ then $q = \frac{m}{n}$ $m, n > 0$. We have from Problem 1 that $x^{\frac{1}{n}} > y^{\frac{1}{n}}$. Also, we know that if z and w are real numbers and $n > 0$ is an integer, then $z^n > w^n$. Hence $x^p = (x^{\frac{1}{n}})^m > (y^{\frac{1}{n}})^n = y^p$. Conversely, suppose $x^q > y^q$. As $q = \frac{m}{n}$ $m, n > 0$ $x^m = (x^q)^n > (y^q)^n = y^m$ But then another application of problem one gives. $\diamond$
- If $x > 1$ then $x^q > x^r$ iff $q > r$ and opposite for $x < 1$. The case $x < 1$ follows from the first case (why?). If $q > r$ then $q - r > 0$. Thus by the previous, $x^{q-r} > 1$ and since $x^r > 0$ $x^q > x^r$. Conversely, we must show that if $x > 1$ and $x^{q-r} > 1$, $a = q - r > 0$. We may assume that $a = \frac{b}{c}$ with $c > 0$. Suppose for a contradiction that $b < 0$ (Note that $b \neq 0$). If $x > 1$ then $1 > \frac{1}{x}$ so that $1 > (\frac{1}{x})^{-b}$, or $1 > x^b$. But then $1 > (x^b)^{\frac{1}{n}}$ by the assumption we made above. That is we have shown that $1 > x^{q-r}$, contrary to the hypothesis. $\diamond$

Problem #6

- A: If $\sum_{i=m}^{\infty} a_i$ converges, this means that the partial sums S_k converge. Thus the sequence $\{S_k\}$ is Cauchy. Fix $\epsilon > 0$. Choose N so large that the sequence $\{S_k\}_{k \geq N}\}$ is ϵ steady. Then, if $p \geq q \geq N$, $\epsilon > |S_p - S_q| = |\sum_{i=q}^{p} a_i|$, where we have used the sum operations proved earlier in the HW, and with this we are done. Conversely, using that $|S_p - S_q| = |\sum_{i=q}^{p} a_i|$, the hypothesis implies that our sequence of partial sums is Cauchy. The BIG THEOREM tells us that $\{S_k\}$ therefore converges to some L. $\diamond$
- B: $|a_n| = |S_n - S_{n-1}|$. Thus $\{S_i\}$ convergent implies $\{S_i\}$ Cauchy. Thus given $\epsilon > 0$, there exists an N such that $|S_p - S_q| < \epsilon$ for $p, q > N$. But then by the remark above, $|a_n| < \epsilon$ if $n > N$. $\diamond$
- C: Suppose $\sum_{i=m}^{\infty} a_i$ is absolutely convergent. Then by A, given $\epsilon > 0$, there exists N such that if $p > q \geq N$, $|S_p - S_q| \leq \sum_{i=q}^{p} |a_i| < \epsilon$, with the first inequality a result of lemma 8. Again using lemma 8, $|S_k| \leq \sum_{i=m}^{n} |a_i| \leq \sum_{i=m}^{\infty} |a_i| k \leq n$. Thus given ϵ, $|\sum_{i=m}^{\infty} a_i \leq |S_k| + \epsilon \leq \sum_{i=m}^{\infty} |a_i| + \epsilon$ if k is large enough. $|\sum_{i=m}^{\infty} a_i \leq \sum_{i=m}^{\infty} |a_i| + \epsilon$ for ϵ arbitrary, and we are done. $\diamond$

Problem # 10 The second statement follows from the first by problem 6, so we prove the $\sum_{n=1}^{\infty} n^q x^n$ is absolutely convergent provided $|x| < 1$. Now, $\limsup |\frac{a_{n+1}}{a_n}| = |x| \limsup |(1 + \frac{1}{n})^q|$. $1 + \frac{1}{n}$ converges to 1, so it suffices to show $(1 + \frac{1}{n})^q$ converges to 1. First suppose that $q = \frac{p}{s} p, s > 0$. We have $1 < (1 + \frac{1}{n})^{\frac{1}{s}} < 1 + \frac{1}{n}$. So if $p = 1$, we apply the squeeze theorem. If not then the above inequalities give $1 < (1 + \frac{1}{n})^{\frac{p}{s}} < (1 + \frac{1}{n})^p$. So the squeeze theorem can be applied if $(1 + \frac{1}{n})^p$ converges to 1. We prove this by induction, the case $p = 0$ obvious. Suppose the assertion is true for $p = k$, then $(1 + \frac{1}{n})^{p+1} = (1 + \frac{1}{n})^p (1 + \frac{1}{n})$. But now limit laws tell us that $\lim_n (1 + \frac{1}{n})^{p+1} = 1$. For the general case of a positive real number α, $1 < (1 + \frac{1}{n})^{\alpha} < (1 + \frac{1}{n})^p$ whenever $\alpha < p$. But then another application of the squeeze theorem gives the result. The proof if $q < 0$ similar, using that $\frac{1}{1 + \frac{1}{n}}$ also converges to one. $\diamond$