1. Solutions to HW3

Problem #8

There are at least two ways to do this problem; a direct induction or the more elegant use of the Euclidean algorithm. We present the latter. We wish to exhibit a bijection between the numbers $\{0, \ldots, nm-1\}$. Given any number $q \leq nm-1$, the Euclidean algorithm gives $q = pm + r$ with $0 \leq r \lneq m$ and this representation is unique. If X has cardinality n and Y has m, given by the bijections f and g respectively, then we will use these to create a bijection h onto $X \times Y$. Note that we take f (g) to have domain $\{0, \ldots, n-1(m-1)\}$.

S. o, let $q \in \{0, \ldots, nm-1\}$, $q = pm + r$. Define $h(q)$ to be $(f(p), g(r)$. Uniqueness of the representation of q gives that this is well defined. h is one to one since $h(q) = h(q')$ implies $f(p) = f(p')$ and $g(r) = g(r')$. As f and g are one to one, $p = p'$ and $r = r'$, so $q = q'$. Finally, given $(x, y) \in X \times Y$, since f and g are onto there exist $t \in \{0, \ldots, n-1\}$ and $z \in \{0, \ldots, m-1\}$ with $f(t) = x$ and $g(z) = y$. So $h(tm + z) = (x, y)$. $\diamond$

Problem # 9

Suppose X and Y are nonempty and for definiteness, X is uncountable. Then there exists a bijection from X into $X \times Y$ given by $x \mapsto (x, y)$ with $y \in Y$ fixed. If $X \times Y$ is countable then a HW problem tells us this image is at most countable, and hence X is at most countable, a contradiction. Therefore, $X \times Y$ is uncountable.
C. onversely, If $X \times Y$ is uncountable, then if X and Y were both at most countable would imply, through various HW problems, that $X \times Y$ would also be at most countable. Therefore either X or Y are at most countable. $\diamond$

Problem # 10

Suppose for a contradiction that that $\mathbb{R} \setminus \mathbb{Q}$ is countable. Then $\mathbb{R}$ is countable, being the union of two countable sets. But this contradicts Cantor's Theorem that $\mathbb{R}$ is uncountable. $\diamond$

2. Solutions to HW # 4

Problem # 5

$\{a_n\}$ is a sequence, increasing in n, that is bounded above by M finite. By the definition of $L = \sup\{a_n\}$ $L \leq M$. In particular, L is finite. Fix $\epsilon \gneq 0$ and consider $L - \epsilon$. By definition of L this is no longer an upper bound, so there exists N such that $L - \epsilon \leq a_N \leq L$. But $\{a_n\}$ is increasing and L is the supremum imply $L - \epsilon \leq a_n \leq L \forall n \geq N$. ϵ was chosen arbitrarily, so our sequence is eventually ϵ close to its sup for any $\epsilon \gneq 0$. $\diamond$

Problem #7 cdef

When doing these types of problems (multiple parts) usually one can use previously proved results to build up to new results.

- Let $A_k = \inf\{a_n\}_{n \geq k}$ and $B_k = \sup\{a_n\}_{n \geq k}$. Then $A_k \leq B_k$, A_k are increasing (why?) and B_k are decreasing in k. Clearly, $A_m = inf\{a_n\} \leq A_k \leq \sup A_k = \liminf\{a_n\}$ and similarly for the B_k's, so we must show $\liminf\{a_n\} \leq \limsup\{a_n\}$. Now for all $n \geq k$ we have $A_k \leq A_n \leq B_n$.

Thus $A_k \le B_n \forall n, k$ (Why?). So first taking the supremum over k in this inequality, $\liminf a_n \le B_m \forall m$ and then taking the infemum over m gives $\liminf a_n \le \limsup a_n$. $\diamond$

- If c is a limit point of the sequence, then $\forall \epsilon \gneq 0$ and $\forall N$ there exists $n \ge N$ such that $A_n - \epsilon \le c \le B_n + \epsilon$ (Why?). This means $\liminf a_n - \epsilon \le c \le \limsup a_n + \epsilon$ (Again, I have skipped a small step, How?). As ϵ is arbitrary, we are done. $\diamond$
- We use (b) for this one. Given $\epsilon \gneq 0$ $\forall N$, there exists $n \ge N$ such that $L^+ - \epsilon \le a_n \le L^+$. Note that since L^+ is finite, $L^+ - \epsilon \ne L^x$. But this says that $\forall \epsilon \gneq 0$ the sequence $\{a_n\}$ is continually ϵ steady. Similarly for L^-. $\diamond$
- If a_n converges to c then we know a_n is Cauchy and hence bounded. Therefore, L^- and L^+ are finite, and hence limit points. But $\{a_n\}$ has only one limit point. So $L^- = c = L^+$.
  C. onversely, $L^- = c = L^+$, then given $\epsilon \gneq 0$ we have the existence of N large enough that $L^- - \epsilon \le a_n \le L^+ + \epsilon$. So $\forall \epsilon \gneq 0$, $\{a_n\}$ is eventually ϵ close to c.