Problem 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that there is a bijection $f : X \to Y$ such that

$$\frac{1}{10} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq 10 d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Show that if X is complete, then Y must also be complete.

[A function $f : X \to Y$ is a *bijection* if it is one-to-one and onto. Equivalently, a function $f : X \to Y$ is a bijection if it has an inverse $f^{-1} : Y \to X$].

Solution: Let $\{y_n\}$ be a Cauchy sequence in Y. We have to show that $\{y_n\}$ converges in Y.

- Step 0: Define the sequence $\{x_n\}$ in X by $x_n = f^{-1}(y_n)$.

  Here we are using the fact that f is invertible, so $f^{-1} : Y \to X$ is well defined.

- Step 1: Since $\{y_n\}$ is Cauchy, $\{x_n\}$ is Cauchy.

  Proof: Let $\varepsilon > 0$. Since $\{y_n\}$ is Cauchy, we can find an $N > 0$ such that $d_Y(y_n, y_m) < \varepsilon/10$ for all $n, m > N$. Since $y_n = f(x_n)$ and $y_m = f(x_m)$, we thus see from the hypothesis that $d_X(x_n, x_m) < \varepsilon$ for all $n, m > N$. Thus x_n is Cauchy.

- Step 2: Since $\{x_n\}$ is Cauchy, $\{x_n\}$ converges.

  This is just because X is complete.

- Step 3: Since $\{x_n\}$ converges, $\{y_n\}$ converges.

  Proof: Let x_n converge to x. Then $d(x_n, x) \to 0$ as $n \to \infty$. From hypothesis, we thus have $d(f(x_n), f(x)) \to 0$ as $n \to \infty$, so $d(y_n, f(x)) \to 0$ as $n \to \infty$. Thus, y_n converges.

Many of you got these steps reversed or otherwise out of order.

Problem 2. Let (X, d) be a metric space, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be continuous functions from X to the real line $\mathbf{R}$. Let $\mathbf{R}^2$ be the plane with the Euclidean metric, and let $h : X \to \mathbf{R}^2$ be the function

$$h(x) = (f(x), g(x)).$$

Show that h is continuous.

Solution A (using sequential definition of continuity): Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ as $n \to \infty$. We have to show that $h(x_n) \to h(x)$ as $n \to \infty$.

Since f, g are continuous, $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} |f(x_n) - f(x)| = \lim_{n\to\infty} |g(x_n) - g(x)| = 0.$$

Squaring both sides and adding, then taking square roots, we get

$$\lim_{n\to\infty} \sqrt{|f(x_n) - f(x)|^2 + |g(x_n) - g(x)|^2} = 0,$$

so (since we are using the Euclidean metric)

$$\lim_{n\to\infty} |h(x_n) - h(x)| = 0.$$

Thus $h(x_n) \to h(x)$ as desired.

Solution B (using epsilon-delta definition of continuity): Let $x \in X$ and $\varepsilon > 0$. We have to find a $\delta > 0$ such that $h(B(x, \delta)) \subset B(h(x), \varepsilon)$.

Since f is continuous, we can find a δ_1 such that $f(B(x, \delta_1)) \subset B(f(x), \varepsilon/2)$. Similarly we can find a δ_2 such that $g(B(x, \delta_2)) \subset B(g(x), \varepsilon/2)$.

Now let δ be the minimum of δ_1 and δ_2. We claim that $h(B(x, \delta)) \subset B(h(x), \varepsilon)$.

To see this, let $y \in B(x, \delta)$. Then $y \in B(x, \delta_1)$ and $y \in B(x, \delta_2)$, so $f(y) \in B(f(x), \varepsilon/2)$ and $g(y) \in B(g(x), \varepsilon/2)$. So $|f(y) - f(x)| < \varepsilon/2$ and $|g(y) - g(x)| < \varepsilon/2$.

Since

$$|h(y) - h(x)| = \sqrt{|f(y) - f(x)|^2 + |g(y) - g(x)|^2}$$

we thus have

$$|h(y) - h(x)| < \sqrt{\varepsilon^2/4 + \varepsilon^2/4} < \varepsilon$$

so $h(y) \in B(h(x), \varepsilon)$ as desired.

One can also use the inverse-image-of-open-sets definition of continuity, but it is somewhat cumbersome.

Problem 3.

Let X be a Banach space, and let $T : X \to X$ be a bounded linear operator on X such that $\|T\| < 1$. Let x_0 be an element of X. Show that there exists a unique $x \in X$ such that

$$x = x_0 + Tx.$$

(Hint: use the contraction principle).

Let $\Phi : X \to X$ denote the map

$$\Phi(x) = x_0 + Tx.$$

The problem can be rephrased as that of showing that Φ has exactly one fixed point. Since X is a Banach space, it is complete, so it suffices to show that Φ is a contraction.

To verify this, we compute:

$$\|\Phi(x) - \Phi(y)\| = \|(x_0 + Tx) - (x_0 + Ty)\| = \|Tx - Ty\|$$

$$= \|T(x - y)\| \leq \|T\|\|x - y\| = c\|x - y\|$$

where $c = \|T\|$. Since c doesn't depend on x or y and $0 \leq c < 1$ by hypothesis, Φ is thus a contraction.

Note: It is also true that T is a contraction, and many of you proved this. However, this fact is not directly helpful to the problem.

Problem 4. Let (X, d) be a metric space, and E be a subset of X. Show that the boundary ∂E of E is closed in X.

[The *boundary* ∂E of E is defined to be the set of all points which are adherent to both E and the complement E^c of E.]

Solution A: Since $\partial E = \overline{E} \cap \overline{E^c}$, and the closure of any set is closed, ∂E is the intersection of two closed sets. Since the intersection of any collection of closed sets is closed, ∂E is therefore closed.

Solution B: Let x be adherent to ∂E. Thus for every $r > 0$, the ball $B(x, r)$ must contain some element, say y, in ∂E. Now define $s = r - d(x, y)$, so $B(y, s)$ is contained in $B(x, r)$. Since y is in the boundary of E, it is adherent to both E and E^c, so $B(y, s)$ contains elements from both E and E^c. Hence $B(x, r)$ also contains elements from both E and E^c.

Problem 5. Let (X, d) be a metric space, and let E be a subset of X. Show that if E is compact, then it must be closed in X.

Solution A (using complete/totally bounded characterization of compactness): Since E is compact, it is complete. Now suppose that $x \in X$ is adherent to E. Then there exists a sequence x_n in E which converges to x. Since convergent sequences are Cauchy, x_n must be a Cauchy sequence. Since E is complete, x_n must converge to a point in E. Since a sequence cannot converge to more than one point, x must be in E. Thus E contains all its adherent points and so it is closed.

Solution B (using convergent subsequence characterization of compactness): Suppose that $x \in X$ is adherent to E. Then there exists a sequence x_n in E which converges to x. Since E is compact, there is a subsequence $x_{n_1}, x_{n_2}, \ldots$ which converges in E. Since x_n converges to x, the subsequence must also converge to x. Since a sequence cannot converge to more than one point, x must be in E. Thus E contains all its adherent points.

Solution C (using open cover characterization of compactness): Suppose that $x \in X$ is adherent to E, but that $x \notin E$. Consider the sets $V_n = \{y \in X : d(x, y) > 1/n\}$ for $n = 1, 2, 3, \ldots$. Each of these sets is open. Since $x \notin E$, the sets V_n cover E, because every element y in E is distinct from x and so we must have $d(x, y) > 1/n$ for at least one integer n. Since E is compact, it can be covered by finitely many V_n:

$$E \subset V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_k}.$$

Let $N = \max(n_1, n_2, \ldots, n_k)$. Clearly

$$V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_k} = \{y \in X : d(x, y) > 1/N\}.$$

Thus for every $y \in E$, $d(x, y) > 1/N$. This contradicts the assumption that x is adherent to E.