Problem 1. Let l^∞ be the space of all bounded sequences of real numbers $(x_n)_{n=1}^\infty$, with the sup norm

$$\|x\|_\infty = \sup_{n=1}^{\infty} |x_n|.$$

Show that $(l^\infty, \|\|_\infty)$ is a Banach space. (You may assume that this space satisfies the conditions for a normed vector space).

Solution. Since we are given that this space is already a normed vector space, the only thing left to verify is that $(l^\infty, \|\|_\infty)$ is complete.

Let $x^1, x^2, \ldots$ be a Cauchy sequence in l^∞. (Note that each element x^n of this sequence is an element of l^∞, so each x^n is itself a sequence, say

$$x^n = (x_1^n, x_2^n, \ldots).$$

That's why I'm using superscripts here instead of subscripts.)

We have to find an element x in l^∞ such that x^n converges to x.

Let $\varepsilon > 0$. Because x^n is a Cauchy sequence, we see that there exists an $N > 0$ such that

$$\|x^n - x^m\|_\infty < \varepsilon$$

for all $n, m > N$. Thus

$$\sup_{k=1}^{\infty} |x_k^n - x_k^m| < \varepsilon$$

for all $n, m > N$. In particular, we have

$$|x_k^n - x_k^m|$$

for all k and all $n, m > N$.

This means that for each k, the sequence

$$x_k^1, x_k^2, \ldots$$

is a Cauchy sequence in $\mathbf{R}$. Since $\mathbf{R}$ is complete, we thus have a limit, call it x_k:

$$\lim_{n\to\infty} x_k^n = x_k.$$

Let x denote the sequence $x = (x_1, x_2, \ldots)$.

We'd like to show that x^n converges to x. Choose an $\varepsilon > 0$. By replacing ε with $\varepsilon/2$ in the previous discussion, we can find an $N > 0$ such that

$$|x_k^n - x_k^m| < \varepsilon/2$$

for all k and all $n, m > N$. Taking limits as $m \to \infty$, we obtain

$$|x_k^n - x_k| \leq \varepsilon/2$$

for all k and all $n > N$. Taking supremum in k, we obtain

$$\sup_{k=1}^{\infty} |x_k^n - x_k| \leq \varepsilon/2$$

for all $n > N$. In other words,

$$\|x^n - x\|_\infty \leq \varepsilon/2 < \varepsilon$$

for all $n > N$. This implies that x^n converges to x, and we are done.

Problem 2. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Prove that there exists a bounded sequence $(b_n)_{n=1}^{\infty}$ such that

$$b_{n-1} + 4b_n + b_{n+1} = a_n \qquad (*)$$

for all $n = 1, 2, \ldots$, where we take b_0 to equal 0. [You may assume the result of Problem 1].

Hint: Use the Contraction Mapping theorem. You may need to rewrite the recurrence (*).

Solution: We can rewrite the recurrence as

$$b_n = \frac{a_n}{4} - \frac{b_{n-1} + b_{n+1}}{4}.$$

Thus we want b to be a fixed point of the operator T defined by

$$(Tb)_n := \frac{a_n}{4} - \frac{b_{n-1} + b_{n+1}}{4}.$$

Note that if b is a bounded sequence, then Tb is automatically a bounded sequence (since we are assuming a is bounded). Thus T is a function from l^∞ to l^∞. To apply the Contraction mapping theorem we now have to verify that T is a contraction on l^∞. In other words, we have to show that

$$\|Tx - Ty\|_\infty \le c\|x - y\|_\infty$$

for some $0 \le c < 1$ and all $x, y \in l^\infty$.

Write $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. We write the left-hand side as

$$\|Tx - Ty\|_\infty = \sup_{n=1}^{\infty} |Tx_n - Ty_n|.$$

Using the definition of T and cancelling, this is

$$\|Tx - Ty\|_\infty = \sup_{n=1}^{\infty} |-\frac{x_{n-1} + x_{n+1}}{4} + \frac{y_{n-1} + y_{n+1}}{4}|.$$

We can re-arrange this as

$$\|Tx - Ty\|_\infty = \frac{1}{4}\sup_{n=1}^{\infty} |-(x_{n-1} - y_{n-1}) - (x_{n+1} - y_{n+1})|.$$

Both terms in parentheses are clearly less than $\|x - y\|_\infty$, so we have

$$\|Tx - Ty\|_\infty \le \frac{2}{4}\|x - y\|_\infty$$

which gives the desired contraction.

Problem 3.

Let $T_1, T_2, \ldots$ be a sequence of continuous linear transformations from a Banach space X to a normed vector space Y. Assume that none of the T_i are identically zero; in other words, for every i there exists a $x \in X$ such that $T_i x \neq 0$. Show that there exists a single $x \in X$ (which does not depend on i) such that $T_i x \neq 0$ for every i.

Hint: use the Baire Category theorem.

Solution: For each i, let S_i denote the set

$$S_i = \{x \in X : T_i x \neq 0.\}$$

Our objective is to find a point $x \in X$ which is not contained in any of the S_i. On the Baire category theorem states that in a complete metric space, the countable union of open dense sets is itself dense (and hence non-empty). Since X is a Banach space, it is a complete metric space, and so we will be done if we can show that each S_i is open and dense.

The open-ness is easy, because S_i is the inverse image under T_i of $\mathbf{R}\backslash\{0\}$, which is an open set, and the inverse image of any open set under a continuous map is open. Now we show that it is dense. This means we need to show that for every ball $B(x, r)$ in X contains at least one point in S_i.

Suppose for contradiction that there was a ball $B(x, r)$ in X which did not contain a point in S_i. In other words, that $T_i y = 0$ for all $y \in B(x, r)$. In particular, we have $T_i x = 0$.

Now let z be any point in X (not necessarily in $B(x, r)$). If we choose N big enough, then the point $x + z/N$ is in $B(x, r)$, so $T_i(x + z/N) = 0$. But we also have $T_i x = 0$. Since T_i is linear, this is only possible if $T_i z = 0$. Since z is arbitrary, this means that T_i is identically zero, a contradiction. Hence the set $T_i x \neq 0$ is dense, and we are done.

Problem 4.

(a) Show that the product of two totally bounded sets is totally bounded.

Solution: Let X, Y be totally bounded sets. We will give $X \times Y$ the Euclidean metric

$$d((x_1, y_1), (x_2, y_2)) = (d(x_1, x_2)^2 + d(y_1, y_2)^2)^{1/2}$$

(all the product metrics are equivalent, so there is no distinction to be made).

Pick an $\varepsilon > 0$. We have to cover $X \times Y$ by finitely many balls of radius ε. Since X is totally bounded, it can be covered by finitely many balls of radius $\varepsilon/10$, say

$$X \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon/10).$$

Similarly we can cover Y by finitely many balls of radius $\varepsilon/10$:

$$Y \subseteq \bigcup_{j=1}^{m} B(y_j, \varepsilon/10).$$

We now claim that $X \times Y$ can be covered by the finite number of balls

$$X \times Y \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B((x_i, y_j), \varepsilon),$$

which will solve the problem.

Pick any $(x, y) \in X \times Y$. Since X is covered by the $B(x_i, \varepsilon/10)$, we can find an i such that $x \in B(x_i, \varepsilon/10)$. Similarly we can find a j such that $y \in B(y_j, \varepsilon/10)$. This means that $d(x, x_i) < \varepsilon/10$ and $d(y, y_j) < \varepsilon/10$. Thus

$$d((x, y), (x_i, y_j)) = (d(x, x_i)^2 + d(y, y_j)^2)^{1/2} < (\varepsilon^2/100 + \varepsilon^2/100)^{1/2} < \varepsilon$$

so (x, y) is in the ball $B((x_i, y_j), \varepsilon)$. This finishes the proof that $X \times Y$ is totally bounded.

(b) Show that every bounded set in $\mathbf{R}^n$ is totally bounded.

Solution: Let $E \subset \mathbf{R}^n$ be a bounded set. Since E is bounded, it is contained in a ball. Since every ball in $\mathbf{R}^n$ is contained in a cube, E must therefore be contained in a cube $I_1 \times I_2 \times \ldots \times I_n$, where all the sides I_j are intervals.

All intervals are totally bounded (for any $\varepsilon > 0$, any interval $[a, b]$ can be covered by finitely many balls of radius ε). Also, from (a) the product of any two totally bounded sets is totally bounded. Thus the cube $I_1 \times I_2 \times \ldots \times I_n$ is totally bounded, and hence E is also totally bounded.

Problem 5. Suppose $f : X \to Y$ is a continuous map from a metric space X to a metric space Y.

(a) Is the inverse image of a closed set under f always closed? Justify your answer.

Solution: Yes. Let E be a closed set in Y. Then the complement E^c is open in Y, hence the inverse image $f^{-1}(E^c)$ is open in X. Now observe that $f^{-1}(E)^c = f^{-1}(E^c)$ (because both sets consist of those points $x \in X$ such that $f(x) \notin E$), so $f^{-1}(E)^c$ is open, which means that $f^{-1}(E)$ is closed.

(b) Is the inverse image of a compact set under f always compact? Justify your answer.

Solution: No. For instance, let $X = Y = \mathbf{R}$, and let f be the constant function $f(x) = 0$. Then $\{0\}$ is compact, but the inverse image of $\{0\}$ is all of $\mathbf{R}$, which is not compact.