Math 115ah, Spring 00, Midterm 1 (Modified)

October 24, 2002

Name:

Student #:

Nickname:

Put down a nickname if you want your score posted.

There are five problems. You have 50 minutes. Do as many problems as you can in this time, skip those which you cannot solve. Each problem is worth 5 points.

Problem 1

Let $\beta := ((1,0),(0,1))$ be the standard ordered basis for $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T(1,2) = (3,2)$$

$$T(1,1) = (1,1).$$

Compute $[T]_\beta^\beta$.

Problem 2

Let $T : V \to W$ be a linear transformation from one vector space V to another vector space W. Let $v_1, \ldots, v_n$ be vectors in V. Assume the span of $v_1, \ldots, v_n$ contains the null space $N(T)$ of T, and assume that the vectors $Tv_1, \ldots, Tv_n$ span W. Prove that the vectors $v_1, \ldots, v_n$ span V. (Hint: take any vector v in V. Then by assumption, Tv is in W and is a linear combination of $Tv_1, \ldots, Tv_n$. Now try to get rid of the T and write v as a linear combination of $v_1, \ldots, v_n$).

Problem 3

Let V_1, V_2, V_3, V_4 be vector spaces such that

$$\dim(V_1) = 8, \ \dim(V_2) = 5, \ \dim(V_3) = 7, \ \dim(V_4) = 6.$$

Let $T_1 : V_1 \to V_2$, $T_2 : V_2 \to V_3$, and $T_3 : V_3 \to V_4$ be linear transformations. Let $T = T_3T_2T_1$ be their composition. Prove that T is not surjective. (Hint: $R(T)$ is a subspace of $R(T_3T_2)$ (why?). How large can the space $R(T_3T_2)$ be, if V_2 is five-dimensional?)

Problem 4

Let V be the vector space over $\mathbb{R}$ consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ with addition and scalar multiplication defined as usual by

$$(f + g)(x) = f(x) + g(x)$$

$$(af)(x) = a(f(x))$$

Let W be the subspace of V consisting of all functions f for which there exist real numbers a, b, c, d such that

$$f(x) = a\sin(x + b) + c\cos(x + d)$$

(You may use without proof that W is a subspace of V and therefore a vector space by itself.) Show that $\{\sin(x), \cos(x)\}$ is a basis for the space W. (Hint: you may find the sine and cosine addition rules

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

to be handy). What is $\dim(W)$?

Problem 5

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

What is the rank and nullity of T?