- Q1. Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be the transformation

$$T(x_1, x_2, x_3, x_4) := (0, x_1, x_2, x_3).$$

- (a) What is the rank and nullity of T?

- (b) Let $\beta := ((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1))$ be the standard ordered basis for T. Compute $[T]_\beta^\beta$, $[T^2]_\beta^\beta$, $[T^3]_\beta^\beta$, and $[T^4]_\beta^\beta$. (Here $T^2 = T \circ T$, $T^3 = T \circ T \circ T$, etc.)

- Q2. Let V denote the space

$$V := \{f \in P_3(\mathbf{R}) : f(0) = f(1) = 0\}.$$

  (a) Show that V is a vector space.

  (b) Find a basis for V. (Hint: if $f(0) = f(1) = 0$, what can one say about the factors of f?)

- Q3. Let V and W be vector spaces, and let $T : V \to W$ be a *one-to-one* linear transformation. Let U be a finite-dimensional subspace of V. Show that the vector space

$$T(U) := \{Tv : v \in U\}$$

  has the same dimension as U. (You may assume without proof that $T(U)$ is a vector space).

- Q4. Let V be a three-dimensional vector space with an ordered basis $\beta := (v_1, v_2, v_3)$. Let γ be the ordered basis $\gamma := ((1,1,0), (1,0,0), (0,0,1))$ of $\mathbf{R}^3$. (You may assume without proof that γ is indeed an ordered basis).

- Let $T : V \to \mathbf{R}^3$ be a linear transformation whose matrix representation $[T]_\beta^\gamma$ is given by

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

  Compute $T(v_1 + 2v_2 + 3v_3)$.

- Q5. Find a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ whose null space $N(T)$ is equal to the z-axis

$$N(T) = \{(0, 0, z) : z \in \mathbf{R}\}$$

and whose range $R(T)$ is equal to the plane

$$R(T) = \{(x, y, z) \in \mathbf{R}^3 : x + y + z = 0\}.$$