(Partial) Solutions to Homework 4

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Q4:

Claim. *If V is a vector space and $T : V \to V$ is a linear map then $T^2 = 0$ iff $R(T) \subset N(T)$.*

Proof: If $R(T) \subset N(T)$ then, since $T(V) = R(T)$ by definition, $T^2(V) = T(T(V)) = T(R(T)) \subset T(N(T)) = \{0\}$, since $T(N(T)) = \{0\}$ by the definition of $N(T)$. Since 0 is clearly an element of $T^2(V)$ ($T^2(0) = 0$), we have that $T^2(V) = \{0\}$. Thus for every $v \in V$, $T^2 v = 0$, i.e. T^2 is the zero map. [Here we have used the fact that $X \subset Y$ implies $T(X) \subset T(Y)$. If this is unclear then let me know, but think about it a little first.]

Conversely, if T^2 is the zero map then, using the same trick, we have that $T^2(V) = T(R(T)) = \{0\}$. Yet then every vector $v \in R(T)$ must have the property that $Tv = 0$. Since $N(T)$ is, by definition, the set of all vectors in V with this property, we find that $R(T) \subset N(T)$. □

Q6:

Claim. *If A is invertible and $AB = 0$ then $B = 0$.*

Proof: We have

$$\begin{aligned} AB &= 0 \\ A^{-1}AB &= A^{-1}0 \\ B &= 0. \end{aligned}$$

□

Q8:

Claim. *The map $T : P_3(\mathbb{R}) \to M_2(\mathbb{R})$ defined by*

$$T(f) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}$$

is injective.

Proof: There is a somewhat simpler proof than Lagrange interpolation if you happen to remember the following fact:

a nonzero polynomial of degree n has at most n zeros.

The matrix $T(f)$ is zero iff the polynomial f has zeros at 0,1,2, and 3–that is four zeros. Because of the basic fact stated above, any polynomial $f \in P_3(\mathbb{R})$ which maps to the zero matrix must therefore be the zero polynomial. □

Q9:

Claim. *Let U, V, W be vector spaces.*

1. *The space U is isomorphic to U.*
2. *If U is isomorphic to V, then V is isomorphic to U.*
3. *If U is isomorphic to V, and V is isomorphic to W, then U is isomorphic to W.*

Proof: 1. Clearly the identity map $I : U \to U$ defined by $Iu = u$ for each $u \in U$ is a bijective (one-to-one and onto) linear map. It follows that U is isomorphic to U.

2. If U is isomorphic to V, there must be an invertible linear map $T : U \to V$ between them. Yet then $T^{-1} : V \to U$ is an invertible linear map, whence V is isomorphic to U.

3. If U is isomorphic to V, and V is isomorphic to W, then there are invertible linear maps $T : U \to V$ and $S : V \to W$. Since we know that $(ST)^{-1} = T^{-1}S^{-1}$ the transformation $ST : U \to W$ is invertible, so U is isomorphic to W. □