Mathematics 115A/3
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Final Examination, Dec 10, 2002

Put down a nickname if you want your score posted.

The test consists of eight problems of varying difficulty and value, adding up to 100 points.

Good luck!

Full name: ______________________

Student ID: ______________________

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Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Problem 6. __________

Problem 7. __________

Problem 8. __________

Total __________

Problem 1. (15 points) Let W be a finite-dimensional real vector space, and let U and V be two subspaces of W. Let $U + V$ be the space

$$U + V := \{u + v : u \in U \text{ and } v \in V\}.$$

You may use without proof the fact that $U + V$ is a subspace of W.

(a) (5 points) Show that $\dim(U + V) \leq \dim(U) + \dim(V)$.

Proof. Since W is finite-dimensional, the subspaces U and V are also finite-dimensional. Suppose that $\dim(U) = n$ and $\dim(V) = m$. Then U has a basis $\{u_1, \ldots, u_n\}$ and V has a basis $\{v_1, \ldots, v_m\}$. Since every $u \in U$ can be written as a linear combination of $u_1, \ldots, u_n$, and every $v \in V$ can be written as a linear combination of $v_1, \ldots, v_m$, we see that every $u + v \in U + V$ can be written as a linear combination of $u_1, \ldots, u_n, v_1, \ldots, v_m$. Thus $U + V$ is spanned by the set $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$, which has at most $n + m$ elements. Thus $\dim(U + V) \leq n + m = \dim(U) + \dim(V)$.

(b) (5 points) Suppose we make the additional assumption that $U \cap V = \{0\}$. Now prove that $\dim(U + V) = \dim(U) + \dim(V)$.

Proof. We use the notation from (a). We need to show that $\dim(U + V) = n + m$; this will happen if we show that $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ not only spans $U + V$ (which we have already shown) but is also linearly independent. Suppose for contradiction that this set was not linearly independent. Then we could find scalars $a_1, \ldots, a_n, b_1, \ldots, b_m$, not all zero, such that

$$a_1 u_1 + \ldots + a_n u_n + b_1 v_1 + \ldots + b_n v_n = 0,$$

which we rewrite as

$$a_1 u_1 + \ldots + a_n u_n = -b_1 v_1 - \ldots - b_n v_n.$$

The left-hand side clearly lies in U (since U is closed under linear combinations), while the right-hand side lies in V. But $U \cap V = \{0\}$, thus both sides are equal to 0. Since $u_1, \ldots, u_n$ are linearly independent, this forces $a_1 = \ldots = a_n = 0$. Since $v_1, \ldots, v_m$ are linearly independent, this forces $b_1 = \ldots = b_m = 0$. But this contradicts the assumption that $a_1, \ldots, a_n, b_1, \ldots, b_m$ were not all zero. Thus $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ are linearly independent and $\dim(U + V) = n + m = \dim(U) + \dim(V)$.

Problem 1 continues on the next page.

Problem 1 continued.

(c) (5 points) Let U and V be two three-dimensional subspaces of $\mathbf{R}^5$. Show that there exists a non-zero vector $v \in \mathbf{R}^5$ which lies in both U and V. (Hint: Use (b) and argue by contradiction).

Suppose for contradiction that there wasn't any non-zero vector v which lay in both U and V. Then $U \cap V$ only contains the zero vector, so by (b) $\dim(U + V) = \dim(U) + \dim(V)$. But $\dim(U) = 3$ and $\dim(V) = 3$ by hypothesis, so $\dim(U + V) = 6$. On the other hand, $U + V$ is a subspace of $\mathbf{R}^5$ which has dimension at most 5. Thus $6 \leq 5$, a contradiction. Thus $U \cap V$ must contain at least one non-zero vector.

Problem 2. (10 points) Let $P_2(\mathbf{R})$ be the space of polynomials of degree at most 2, with real coefficients. We give $P_2(\mathbf{R})$ the inner product

$$\langle f, g\rangle := \int_0^1 f(x)g(x)\ dx.$$

You may use without proof the fact that this is indeed an inner product for $P_2(\mathbf{R})$.

(a) (5 points) Find an orthonormal basis for $P_2(\mathbf{R})$.

Applying the Gram-Schmidt orthogonalization process to $v_1 = 1, v_2 = x, v_3 = x^2$ yields

$$w_1 = v_1 = 1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = x - \frac{1/2}{1} 1 = x - 1/2$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$= x^2 - \frac{1/3}{1} 1 - \frac{1/12}{1/12}(x - 1/2) = x^2 - x + 1/6.$$

Normalizing this yields

$$w_1' = w_1/\|w_1\| = 1$$

$$w_2' = w_2/\|w_2\| = \sqrt{12}(x - 1/2)$$

$$w_3' = w_3/\|w_3\| = \sqrt{180}(x^2 - x + 1/6).$$

(b) (5 points) Find a basis for $\text{span}(1, x)^{\perp}$.

Since $\text{span}(1, x)$ is spanned by w_1' and w_2', and w_1', w_2', w_3' is orthonormal, then $\text{span}(1, x)^{\perp}$ is spanned by $w_3' = \sqrt{180}(x^2 - x + 1/6)$. (It is also an acceptable answer to say that $\text{span}(1, x)^{\perp}$ is spanned by $w_3 = x^2 - x + 1/6$).

Problem 3. (15 points) Let $P_3(\mathbf{R})$ be the space of polynomials of degree at most 3, with real coefficients. Let $T : P_3(\mathbf{R}) \to P_3(\mathbf{R})$ be the linear transformation

$$Tf := \frac{df}{dx},$$

thus for instance $T(x^3 + 2x) = 3x^2 + 2$. You may use without proof the fact that T is indeed a linear transformation. Let $\beta := (1, x, x^2, x^3)$ be the standard basis for $P_3(\mathbf{R})$.

(a) (5 points) Compute the matrix $[T]_\beta^\beta$.

Answer:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) (3 points) Compute the characteristic polynomial of $[T]_\beta^\beta$.

Answer:

$$\lambda^4$$

(c) (5 points) What are the eigenvalues and eigenvectors of T? (Warning: the eigenvectors of T are related to, but not quite the same as, the eigenvectors of $[T]_\beta^\beta$.

0 is the only eigenvalue (this is the only root of the characteristic polynomial). The eigenvectors are those elements $f \in P_3(\mathbf{R})$ such that $Tf = 0f$, i.e. $\frac{df}{dx} = 0$, i.e. $f = c$ for some constant scalar c. (The eigenvectors of $[T]_\beta^\beta$ are $\begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}$, which of course are the co-ordinate vectors of c in the basis β.)

Problem 3 continues on the next page.

Problem 3 continued.

(d) (2 points) Is T diagonalizable? Explain your reasoning.

Answer: The eigenvectors do not span $P_3(\mathbf{R})$, so T is not diagonalizable. (The fact that T splits is not enough to justify diagonalizability; the fact that T has repeated roots is not enough to prohibit diagonalizability).

Problem 4. (15 points) This question is concerned with the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T(x, y, z, w) := (x + y + z, y + 2z + 3w, x - z - 2w).$$

You may use without proof the fact that T is a linear transformation.

(a) (5 points) What is the nullity of T?

Answer: 1. See (b).

(b) (5 points) Find a basis for the null space. (This basis does *not* need to be orthogonal or orthonormal).

Answer: The null space consists of all those vectors (x, y, z, w) such that $x + y + z = 0$, $y + 2z + 3w = 0$, and $x - z - 2w = 0$. After some Gaussian elimination we see that this reduces to $w = 0$, $y = -2z$, and $x = z$, thus $(x, y, z, w) = (z, -2z, z, 0) = z(1, -2, 1, 0)$. Thus $(1, -2, 1, 0)$ is a basis for the null space.

(c) (5 points) Find a basis for the range. (This basis does *not* need to be orthogonal or orthonormal).

Since the nullity is 1, the rank is 3. Since T maps into $\mathbf{R}^3$, the range of T must therefore be all of $\mathbf{R}^3$. Thus any basis of $\mathbf{R}^3$ will work for this question, e.g. $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

Problem 5. (10 points) Let V be a real vector space, and let $T : V \to V$ be a linear transformation such that $T^2 = T$. Let $R(T)$ be the range of T and let $N(T)$ be the null space of T.

(a) (5 points) Prove that $R(T) \cap N(T) = \{0\}$.

Proof. Clearly 0 lies in both $R(T)$ and $N(T)$, since they are subspaces of V. Now we have to show that 0 is the only element of $R(T) \cap N(T)$. Suppose for contradiction that there was a non-zero element $v \in R(T) \cap N(T)$. Since $v \in N(T)$, we see that $Tv = 0$. Since $v \in R(T)$, we see that $v = Tw$ for some $w \in V$. Putting the two equations together, we see that $T^2 w = 0$. But $T^2 = T$, so $Tw = 0$. But $Tw = v$, hence $v = 0$, contradicting our assumption that v was non-zero. Thus $R(T) \cap N(T)$ contains the 0 vector but nothing else.

(b) (5 points) Let $R(T) + N(T)$ denote the space

$$R(T) + N(T) := \{x + y : x \in R(T) \text{ and } y \in N(T)\}.$$

Show that $R(T) + N(T) = V$. (**Hint:** First show that for any vector $v \in V$, the vector $v - Tv$ lies in the null space $N(T)$).

Proof. Clearly $R(T) + N(T)$ is contained in V since V is closed under addition. Now conversely we need to show that V is contained in $R(T) + N(T)$. So let v be any vector in V. Since $T(v - Tv) = Tv - T^2 v = Tv - Tv = 0$, we see that $v - Tv$ lies in the null space of T. Also, Tv lies in the range of T by definition. Thus $v = Tv + (v - Tv)$ lies in $R(T) + N(T)$ as desired.

Problem 6. (15 points) Let A be the matrix

$$A := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(a) (5 points) Find a complex invertible matrix Q and a complex diagonal matrix D such that $A = QDQ^{-1}$. (Hint: A has -1 as one of its eigenvalues).

Answer: The eigenvalues of A are $+i, -i, -1$ with eigenvectors $\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, thus one possible choice of D and Q are

$$D = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Other answers are also possible (e.g. by swapping some of the rows, or by multiplying some of the eigenvectors by a scalar).

Problem 6 continues on the next page.

Problem 6 continued.

(b) (5 points) Find three elementary matrices E_1, E_2, E_3 such that $A = E_1E_2E_3$. (Note: this problem is not directly related to (a)).

Answer: The idea is to convert A to the identity matrix using three elementary row or column operations, and then work out what you did in terms of matrices. One sample factorization is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(c) (5 points) Compute A^{-1}, by any means you wish.

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Problem 7. (10 points) Let f, g be continuous, complex-valued functions on $[-1, 1]$ such that $\int_{-1}^{1} |f(x)|^2 \, dx = 9$ and $\int_{-1}^{1} |g(x)|^2 \, dx = 16$.

(a) (5 points) What possible values can $\int_{-1}^{1} f(x)\overline{g(x)} \, dx$ take? Explain your reasoning.

Answer: We use the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)\overline{g(x)} \, dx$ on $C([-1, 1]; \mathbf{C})$. By hypothesis, $\|f\| = \sqrt{\langle f, f \rangle} = 3$ and $\|g\| = \sqrt{\langle g, g \rangle} = 4$, so by Cauchy-Schwarz $|\langle f, g \rangle| \leq \|f\| \|g\| = 12$. Thus $\langle f, g \rangle$ can take any complex number with magnitude at most 12.

(b) (5 points) What possible values can $\int_{-1}^{1} |f(x) + g(x)|^2 \, dx$ take? Explain your reasoning.

Answer: By the triangle inequality we have $\|f + g\| \leq \|f\| + \|g\| = 4 + 3 = 7$, and so $\int_{-1}^{1} |f(x) + g(x)|^2 = \|f + g\|^2 \leq 49$. But also $\|f + g\| \geq \|g\| - \|f\| = 4 - 3 = 1$, so $\int_{-1}^{1} |f(x) + g(x)|^2 = \|f + g\|^2 \geq 1$. In fact $\int_{-1}^{1} |f(x) + g(x)|^2$ can equal any real number between 1 and 49 inclusive.

Problem 8. (10 points) Find a 2×2 matrix A with real entries which has trace 5, determinant 6, and has $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as one of its eigenvectors. (Hint: First work out what the characteristic polynomial of A must be. There are several possible answers to this question; you only have to supply one of them.)

Answer: The characteristic polynomial is $\lambda^2 - 5\lambda + 6$, so the eigenvalues are 2 and 3. One can either assign the eigenvalue 2 to the first eigenvector or to the second eigenvector. In the first case one gets the matrix

$$\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$$

and in the second case one gets the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$