

**Mathematics 115A/3**  
**Terence Tao**  
**Final Examination, Dec 10, 2002**

Put down a nickname if you want your score posted.

The test consists of eight problems of varying difficulty and value, adding up to 100 points.

Good luck!

**Full name:** \_\_\_\_\_

**Student ID:** \_\_\_\_\_

**Nickname:** \_\_\_\_\_

**Signature:** \_\_\_\_\_

Problem 1. \_\_\_\_\_

Problem 2. \_\_\_\_\_

Problem 3. \_\_\_\_\_

Problem 4. \_\_\_\_\_

Problem 5. \_\_\_\_\_

Problem 6. \_\_\_\_\_

Problem 7. \_\_\_\_\_

Problem 8. \_\_\_\_\_

Total \_\_\_\_\_

**Problem 1.** (15 points) Let  $W$  be a finite-dimensional real vector space, and let  $U$  and  $V$  be two subspaces of  $W$ . Let  $U + V$  be the space

$$U + V := \{u + v : u \in U \text{ and } v \in V\}.$$

You may use without proof the fact that  $U + V$  is a subspace of  $W$ .

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(a) (5 points) Show that  $\dim(U + V) \leq \dim(U) + \dim(V)$ .

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(b) (5 points) Suppose we make the additional assumption that  $U \cap V = \{0\}$ . Now prove that  $\dim(U + V) = \dim(U) + \dim(V)$ .

Problem 1 continued.

(c) (5 points) Let  $U$  and  $V$  be two three-dimensional subspaces of  $\mathbf{R}^5$ . Show that there exists a non-zero vector  $v \in \mathbf{R}^5$  which lies in both  $U$  and  $V$ . (Hint: Use (b) and argue by contradiction).

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**Problem 2.** (10 points) Let  $P_2(\mathbf{R})$  be the space of polynomials of degree at most 2, with real coefficients. We give  $P_2(\mathbf{R})$  the inner product

$$\langle f, g \rangle := \int_0^1 f(x)g(x) dx.$$

You may use without proof the fact that this is indeed an inner product for  $P_2(\mathbf{R})$ .

(a) (5 points) Find an orthonormal basis for  $P_2(\mathbf{R})$ .

Ans.

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(b) (5 points) Find a basis for  $\text{span}(1, x)^\perp$ .

Ans.

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**Problem 3.** (15 points) Let  $P_3(\mathbf{R})$  be the space of polynomials of degree at most 3, with real coefficients. Let  $T : P_3(\mathbf{R}) \rightarrow P_3(\mathbf{R})$  be the linear transformation

$$Tf := \frac{df}{dx},$$

thus for instance  $T(x^3 + 2x) = 3x^2 + 2$ . You may use without proof the fact that  $T$  is indeed a linear transformation. Let  $\beta := (1, x, x^2, x^3)$  be the standard basis for  $P_3(\mathbf{R})$ .

(a) (5 points) Compute the matrix  $[T]_\beta^\beta$ .

Ans.

(b) (3 points) Compute the characteristic polynomial of  $[T]_\beta^\beta$ .

Ans.

(c) (5 points) What are the eigenvalues and eigenvectors of  $T$ ? (Warning: the eigenvectors of  $T$  are related to, but not quite the same as, the eigenvectors of  $[T]_\beta^\beta$ .)

Ans.

Problem 3 continues on the next page.

Problem 3 continued.

(d) (2 points) Is  $T$  diagonalizable? Explain your reasoning.

**Problem 4.** (15 points) This question is concerned with the linear transformation  $T : \mathbf{R}^4 \rightarrow \mathbf{R}^3$  defined by

$$T(x, y, z, w) := (x + y + z, y + 2z + 3w, x - z - 2w).$$

You may use without proof the fact that  $T$  is a linear transformation.

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(a) (5 points) What is the nullity of  $T$ ?

Ans.

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(b) (5 points) Find a basis for the null space. (This basis does *not* need to be orthogonal or orthonormal).

Ans.

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(c) (5 points) Find a basis for the range. (This basis does *not* need to be orthogonal or orthonormal).

Ans.

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**Problem 5.** (10 points) Let  $V$  be a real vector space, and let  $T : V \rightarrow V$  be a linear transformation such that  $T^2 = T$ . Let  $R(T)$  be the range of  $T$  and let  $N(T)$  be the null space of  $T$ .

(a) (5 points) Prove that  $R(T) \cap N(T) = \{0\}$ .

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(b) (5 points) Let  $R(T) + N(T)$  denote the space

$$R(T) + N(T) := \{x + y : x \in R(T) \text{ and } y \in N(T)\}.$$

Show that  $R(T) + N(T) = V$ . (**Hint:** First show that for any vector  $v \in V$ , the vector  $v - Tv$  lies in the null space  $N(T)$ ).



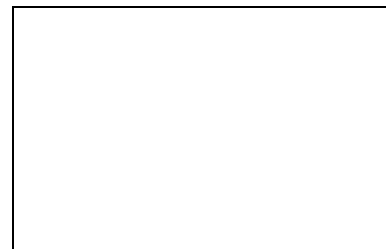
**Problem 6.** (15 points) Let  $A$  be the matrix

$$A := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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(a) (5 points) Find a complex invertible matrix  $Q$  and a complex diagonal matrix  $D$  such that  $A = QDQ^{-1}$ . (Hint:  $A$  has  $-1$  as one of its eigenvalues).

Ans.



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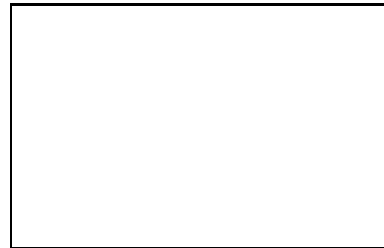
Problem 6 continues on the next page.

Problem 6 continued.

(b) (5 points) Find three elementary matrices  $E_1, E_2, E_3$  such that  $A = E_1 E_2 E_3$ . (Note: this problem is not directly related to (a)).

Ans.

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(c) (5 points) Compute  $A^{-1}$ , by any means you wish.

Ans.

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**Problem 7.** (10 points) Let  $f, g$  be continuous, complex-valued functions on  $[-1, 1]$  such that  $\int_{-1}^1 |f(x)|^2 dx = 9$  and  $\int_{-1}^1 |g(x)|^2 dx = 16$ .

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(a) (5 points) What possible values can  $\int_{-1}^1 f(x)\overline{g(x)} dx$  take? Explain your reasoning.

Ans.

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(b) (5 points) What possible values can  $\int_{-1}^1 |f(x) + g(x)|^2 dx$  take? Explain your reasoning.

Ans.

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**Problem 8.** (10 points) Find a  $2 \times 2$  matrix  $A$  with real entries which has trace 5, determinant 6, and has  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  as one of its eigenvectors. (Hint: First work out what the characteristic polynomial of  $A$  must be. There are several possible answers to this question; you only have to supply one of them.)

Ans.

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