Mathematics 115A/3
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Final Examination, Dec 10, 2002

Put down a nickname if you want your score posted.

The test consists of eight problems of varying difficulty and value, adding up to 100 points.

Good luck!

Full name: ______________________

Student ID: ______________________

Nickname: ______________________

Signature: ______________________

Problem 1. ____________

Problem 2. ____________

Problem 3. ____________

Problem 4. ____________

Problem 5. ____________

Problem 6. ____________

Problem 7. ____________

Problem 8. ____________

Total ____________

Problem 1. (15 points) Let W be a finite-dimensional real vector space, and let U and V be two subspaces of W. Let $U + V$ be the space

$$U + V := \{u + v : u \in U \text{ and } v \in V\}.$$

You may use without proof the fact that $U + V$ is a subspace of W.

(a) (5 points) Show that $\dim(U + V) \leq \dim(U) + \dim(V)$.

(b) (5 points) Suppose we make the additional assumption that $U \cap V = \{0\}$. Now prove that $\dim(U + V) = \dim(U) + \dim(V)$.

Problem 1 continues on the next page.

Problem 1 continued.

(c) (5 points) Let U and V be two three-dimensional subspaces of $\mathbf{R}^5$. Show that there exists a non-zero vector $v \in \mathbf{R}^5$ which lies in both U and V. (Hint: Use (b) and argue by contradiction).

Problem 2. (10 points) Let $P_2(\mathbf{R})$ be the space of polynomials of degree at most 2, with real coefficients. We give $P_2(\mathbf{R})$ the inner product

$$\langle f, g \rangle := \int_0^1 f(x)g(x)\, dx.$$

You may use without proof the fact that this is indeed an inner product for $P_2(\mathbf{R})$.

(a) (5 points) Find an orthonormal basis for $P_2(\mathbf{R})$.

Ans.

(b) (5 points) Find a basis for $\operatorname{span}(1, x)^{\perp}$.

Ans.

Problem 3. (15 points) Let $P_3(\mathbf{R})$ be the space of polynomials of degree at most 3, with real coefficients. Let $T : P_3(\mathbf{R}) \to P_3(\mathbf{R})$ be the linear transformation

$$Tf := \frac{df}{dx},$$

thus for instance $T(x^3 + 2x) = 3x^2 + 2$. You may use without proof the fact that T is indeed a linear transformation. Let $\beta := (1, x, x^2, x^3)$ be the standard basis for $P_3(\mathbf{R})$.

(a) (5 points) Compute the matrix $[T]_\beta^\beta$.

Ans.

(b) (3 points) Compute the characteristic polynomial of $[T]_\beta^\beta$.

Ans.

(c) (5 points) What are the eigenvalues and eigenvectors of T? (Warning: the eigenvectors of T are related to, but not quite the same as, the eigenvectors of $[T]_\beta^\beta$.

Ans.

Problem 3 continues on the next page.

Problem 3 continued.

(d) (2 points) Is T diagonalizable? Explain your reasoning.

Problem 4. (15 points) This question is concerned with the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T(x, y, z, w) := (x + y + z, y + 2z + 3w, x - z - 2w).$$

You may use without proof the fact that T is a linear transformation.

(a) (5 points) What is the nullity of T?

Ans.

(b) (5 points) Find a basis for the null space. (This basis does *not* need to be orthogonal or orthonormal).

Ans.

(c) (5 points) Find a basis for the range. (This basis does *not* need to be orthogonal or orthonormal).

Ans.

Problem 5. (10 points) Let V be a real vector space, and let $T : V \to V$ be a linear transformation such that $T^2 = T$. Let $R(T)$ be the range of T and let $N(T)$ be the null space of T.

(a) (5 points) Prove that $R(T) \cap N(T) = \{0\}$.

(b) (5 points) Let $R(T) + N(T)$ denote the space

$$R(T) + N(T) := \{x + y : x \in R(T) \text{ and } y \in N(T)\}.$$

Show that $R(T) + N(T) = V$. (**Hint:** First show that for any vector $v \in V$, the vector $v - Tv$ lies in the null space $N(T)$).

Problem 6. (15 points) Let A be the matrix

$$A := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(a) (5 points) Find a complex invertible matrix Q and a complex diagonal matrix D such that $A = QDQ^{-1}$. (Hint: A has -1 as one of its eigenvalues).

Ans.

Problem 6 continues on the next page.

Problem 6 continued.

(b) (5 points) Find three elementary matrices E_1, E_2, E_3 such that $A = E_1E_2E_3$. (Note: this problem is not directly related to (a)).

Ans.

(c) (5 points) Compute A^{-1}, by any means you wish.

Ans.

Problem 7. (10 points) Let f, g be continuous, complex-valued functions on $[-1, 1]$ such that $\int_{-1}^{1} |f(x)|^2 \, dx = 9$ and $\int_{-1}^{1} |g(x)|^2 \, dx = 16$.

(a) (5 points) What possible values can $\int_{-1}^{1} f(x)\overline{g(x)} \, dx$ take? Explain your reasoning.

Ans.

(b) (5 points) What possible values can $\int_{-1}^{1} |f(x) + g(x)|^2 \, dx$ take? Explain your reasoning.

Ans.

Problem 8. (10 points) Find a 2×2 matrix A with real entries which has trace 5, determinant 6, and has $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as one of its eigenvectors. (Hint: First work out what the characteristic polynomial of A must be. There are several possible answers to this question; you only have to supply one of them.)

Ans.