Assignment 7 Due November 21 Covers: Sections 4.4; 5.1-5.2

- Q1. Do Question 1(acdefgijk) of Question 1 of Section 4.4 of the textbook.
- Q2. Do Question 4(g) of Section 4.4 of the textbook.
- Q3. Do Question 2(a) of Section 5.1 of the textbook.
- Q4. Do Question 3(a) of Section 5.1 of the textbook. (Treat any occurrence of F as if it were $\mathbf{R}$ instead).
- Q5. Do Question 8 of Section 5.1 of the textbook. (You may assume that $T : V \to V$ is a linear transformation from some finite-dimensional vector space V to itself; this is what it means for T to be a linear transformation "on V).
- Q6*. Do Question 11 of Section 5.1 of the textbook.
- Q7. Do Question 15 of Section 5.1 of the textbook.
- Q8. Do Question 3(bf) of Section 5.2 of the textbook.
- Q9. Let A and B be similar $n \times n$ matrices. Show that A and B have the same set of eigenvalues (i.e. every eigenvalue of A is also an eigenvalue of B and vice versa).
- Q10*. For this question, the field of scalars will be the complex numbers $\mathbf{C} := \{x + yi : x, y \in \mathbf{R}\}$ instead of the reals (i.e. all matrices, etc. are allowed to have complex entries). Let θ be a real number, and let A be the 2×2 rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- (a) Show that A has eigenvalues $e^{i\theta}$ and $e^{-i\theta}$. (You may use Eulers formula $e^{i\theta} = \cos\theta + i\sin\theta$). What are the eigenvectors corresponding to $e^{i\theta}$ and $e^{-i\theta}$?

- (b) Write $A = QDQ^{-1}$ for some invertible matrix Q and diagonal matrix D (note that Q and D may have complex entries. Also, there are several possible answers to this question; you only need to give one of them).

- (c) Let $n \geq 1$ be an integer. Prove that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}.$$

  (You may find the formulae $(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$ and $(e^{-i\theta})^n = e^{-in\theta} = \cos n\theta - i \sin n\theta$ to be useful).

- (d) Can you explain why the operator $L_A : \mathbf{R}^2 \to \mathbf{R}^2$ corresponds to an anti-clockwise rotation of the plane $\mathbf{R}^2$ by angle θ?

- (e) Based on (d), can you think of a geometrical interpretation of the result proven in (c)?