(Partial) Solutions to Homework 6

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Q4: I just thought I would point out the following trick, which might some day make your lives easier (maybe). In parts (d) and (e), you found that the transformations were represented by matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

so that one is just like the other up to a permutation of rows. Specifically,

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} A.$$

Thus we find that B^{-1} is given by

$$\begin{aligned} B^{-1} &= A^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \\ &= A^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

a calculation rather easier than repeating the applications of elementary row operations.

Q7:

Claim. *Let U, V, and W be finite dimensional vector spaces, and let $S : V \to W$ and $T : U \to V$ be linear transformations.*

1. $\text{rank}(ST) \leq \text{rank}(S)$
2. $\text{rank}(ST) \leq \text{rank}(T)$
3. $\text{nullity}(ST) \geq \text{nullity}(T)$

Proof: 1. Since $T(U) \subset V$, we have that $S(T(U)) \subset S(V)$, whence

$$\begin{aligned} \dim S(T(U)) &\leq \dim S(V) \\ \dim ST(U) &\leq \dim S(V) \\ \text{rank}(ST) &\leq \text{rank}(S). \end{aligned}$$

2. By the dimension theorem, letting $S|_{T(U)}$ denote the restriction of S to the range of T,

$$\begin{aligned}\operatorname{rank}(S|_{T(U)}) + \operatorname{nullity}(S|_{T(U)}) &= \dim(T(U)) \\ \operatorname{rank}(ST) + \operatorname{nullity}(S|_{T(U)}) &= \operatorname{rank}(T) \\ \operatorname{rank}(ST) &\leq \operatorname{rank}(T),\end{aligned}$$

where in the last line we have used the fact that $\operatorname{nullity}(S|_{T(U)}) \geq 0$.

3. By the dimension theorem,

$$\operatorname{rank}(ST) + \operatorname{nullity}(ST) = \dim U \qquad \operatorname{rank}(T) + \operatorname{nullity}(T) = \dim U.$$

Thus we have

$$\begin{aligned}\operatorname{rank}(ST) + \operatorname{nullity}(ST) &= \operatorname{rank}(T) + \operatorname{nullity}(T) \\ \operatorname{nullity}(ST) &\geq \operatorname{nullity}(T),\end{aligned}$$

since $\operatorname{rank}(ST) \leq \operatorname{rank}(T)$. (If you aren't sure you believe this, consider the equation $\operatorname{nullity}(T) - \operatorname{nullity}(ST) = \operatorname{rank}(ST) - \operatorname{rank}(T) \leq 0$). □

Q8:

Claim. *If A and B are $n \times n$ matrices then $(AB)^t = B^t A^t$.*

Proof: Let us write $A = (A_{ij})$ and $B = (B_{ij})$. Then we have

$$\begin{aligned}(AB)_{ij} &= \sum_{k=1}^{n} A_{ik} B_{kj} \\ (AB)^t_{ij} &= (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} \\ (AB)^t_{ij} &= \sum_{k=1}^{n} A^t_{kj} B^t_{ik} \\ (AB)^t_{ij} &= \sum_{j=1}^{n} B^t_{ik} A^t_{kj} \\ (AB)^t_{ij} &= (B^t A^t)_{ij},\end{aligned}$$

which yields the desired inequality. □

Q9:

Claim. *If A is an invertible $n \times n$ matrix then* $\det(A^{-1}) = 1/\det A$.

Proof: By the multiplicative property of determinants, we have $1 = \det I_n = \det(AA^{-1}) = \det A \det(A^{-1})$, i.e.$\det A \det(A^{-1}) = 1$. Thus $\det(A^{-1}) = 1/\det A$. □