(Partial) Solutions to Homework 2

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Q3:

Claim. *1. If T is linear then $T(0) = 0$.*

2. A map T is linear iff $T(\alpha x + y) = \alpha T(x) + T(y)$ for all $x, y \in V$ and $\alpha \in F$.

3. A map T is linear iff for $x_1, \ldots, x_n \in V$ and $a_1, \ldots, a_n \in F$ we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i). \tag{1}$$

Proof: 1. If T is linear then for any $v \in V$

$$T(0) = T(0\dot{v}) = 0T(v) = 0.$$

2. If T is linear then

$$T(\alpha x + y) = T(\alpha x) + T(y) = \alpha T(x) + T(y)$$

for all $x, y \in V$ and $\alpha \in F$

If $T(\alpha x + y) = \alpha T(x) + T(y)$ for all $x, y \in V$ and $c \in F$ then picking $\alpha = 1$ gives

$$T(x + y) = T(x) + T(y)$$

for all $x, y \in V$. Similarly, if we pick $y = 0$ then we find that

$$T(\alpha x) = \alpha T(x)$$

for all $x, \in V$ and $\alpha \in F$, where we note that $T(0) = 0$ since for $\alpha = 1$ and $x = y = 0$ we have

$$\begin{aligned} T(0) = T(\alpha x + y) &= T(0) + T(0) \\ 0 &= T(0). \end{aligned}$$

3. If equation (1) holds then choosing $n = 2$, $a_2 = 1$, and denoting $a_1 = \alpha$, $x_1 = x$, and $x_2 = y$ gives

$$\begin{aligned} T\left(\sum_{i=1}^{2} a_i x_i\right) &= \sum_{i=1}^{2} a_i T(x_i) \\ T(\alpha x + y) &= \alpha T(x) + T(y), \end{aligned}$$

which implies that T is linear by (2).

Suppose that T is linear. We will show that the equation (1) holds by induction. For $n = 2$, we have

$$\begin{aligned} T\left(\sum_{i=1}^{2} a_i x_i\right) &= T(a_1 x + a_2 y) \\ &= T(a_1 x) + T(a_2 y) \\ &= a_1 T(x) + a_2 T(y) \\ T\left(\sum_{i=1}^{2} a_i x_i\right) &= \sum_{i=1}^{2} a_i T(x_i). \end{aligned}$$

If we know that the equation holds for $n = k$, we find that

$$\begin{aligned} T\left(\sum_{i=1}^{k+1} a_i x_i\right) &= T\left(\sum_{i=1}^{k} a_i x_i + a_{k+1} x_{k+1}\right) \\ &= T\left(\sum_{i=1}^{k} a_i x_i\right) + T(a_{k+1} x_{k+1}) \\ &= \sum_{i=1}^{k} a_i T(x_i) + a_{k+1} T(x_{k+1}) \\ T\left(\sum_{i=1}^{k+1} a_i x_i\right) &= \sum_{i=1}^{k+1} a_i T(x_i), \end{aligned}$$

which is exactly equation (1) for $n = k + 1$. It follows from induction that (1) holds for all $n \in \mathbb{N}$. □

Q6:

Claim. *Let V be a vector space, and let A, B be two subsets of V. If B spans V and $\langle A \rangle$ contains B, then A spans V.*

Proof: If $B \subset \langle A \rangle$ then for every vector $b \in B$ we have an expansion

$$b = \sum_{a \in A} \alpha_{ba} a$$

for some scalars $\alpha_{ba} \in F$ (all but finitely many are zero for each fixed index b). Since B spans V, we have, for any $x \in V$ an expansion of the form

$$x = \sum_{b \in B} \beta_b b,$$

where only finitely many of the $\beta_b \in F$ are nonzero. Then

$$\begin{aligned} x &= \sum_{b \in B} \beta_b b \\ &= \sum_{b \in B} \beta_b \left(\sum_{a \in A} \alpha_{ba} a\right) \\ x &= \sum_{a \in A} \left(\sum_{b \in B} \beta_b \alpha_{ba}\right) a. \end{aligned}$$

so it follows that A spans V. □

Q7:

Claim. *If V is a vector space which is spanned by a finite set S of n elements then V is finite dimensional, with dimension less than or equal to n.*

Proof: We construct, algorithmically, a maximal linearly independent subset of S.

Choose some nonzero vector $s_1 \in S$. If $\langle s_1 \rangle \neq \langle S \rangle$ then we may find some nonzero element $s_2 \in S \backslash \{s_1\}$ such that $s_2 \notin \langle s_1 \rangle$. Not that s_1 and s_2 are linearly independent. Continuing in this fashion, if $\langle s_1, s_2 \rangle \neq \langle S \rangle$ then we may find a nonzero vector $s_3 \in S \setminus \{s_1, s_2\}$ such that $s_3 \notin \langle s_1, s_2 \rangle$. Now, by construction s_3 is independent of $\{s_1, s_2\}$, and $\{s_1, s_2\}$ is linearly independent by construction, so $\{s_1, s_2, s_3\}$ is a linearly independent set. Inductively, at the kth step we have k linearly independent vectors $\{s_1, s_2, \ldots, s_k\}$. If $\langle s_1, s_2, \ldots, s_k \rangle \neq \langle S \rangle$ then we can find a nonzero vector $s_{k+1} \in S \setminus \{s_1, s_2, \ldots, s_k\}$. Now by construction s_{k+1} is independent of $\{s_1, s_2, \ldots, s_k\}$; since this latter is a linearly independent set, $\{s_1, s_2, \ldots, s_{k+1}\}$ is a linearly independent subset of S.

Since the number of elements in the set S is finite, this process will terminate with some linearly independent subset $\{s_1, \ldots, s_m\}$ of S. Some examination of this algorithm reveals that the only reason that this algorithm might terminate is if $\langle s_1, \ldots, s_m \rangle = \langle S \rangle$. Since S spans V, we have a linearly independent subset of S spanning V. In particular, $m \leq n$, so $\dim V = m \leq n$. □

Q8:

Claim. *The space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$ is infinite dimensional.*

Proof: (Proof 1): Suppose, for contradiction, that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ has finite dimension n. Clearly the polynomials of degree less than or equal to n, $P_n(\mathbb{R})$, are contained in $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Yet then the basis $\{1, x, \ldots, x^n\}$ is a linearly independent set of size $n+1$, contradicting that $\dim \mathcal{F} = n$. Thus $\mathcal{F}(\mathbb{R}, \mathbb{R})$ cannot be finite dimensional.

(Proof 2): Clearly the spaces $P_n(\mathbb{R})$ are contained in $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Thus the linearly independent set $\{1, x, \ldots, x^n\}$ is contained in $\mathcal{F}(\mathbb{R}, \mathbb{R})$, whence $\dim \mathcal{F} > n$. Since the integer n is arbitrary, it follows that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ cannot have finite dimension. □