ABSTRACT. This paper is a continuation of [35], in which new bilinear estimates for surfaces in $\mathbf{R}^3$ were proven. We give a concrete improvement to the square function estimate of Mockenhaupt [20]. We apply these estimates to give new progress on several open problems concerning the wave and Schrödinger equation in $\mathbf{R}^{2+1}$, and convolution with curves in $\mathbf{R}^3$.

A BILINEAR APPROACH TO CONE MULTIPLIERS II. APPLICATIONS.

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1. Introduction

Let S_1, S_2 two smooth compact hypersurfaces with boundary in $\mathbf{R}^3$, with Lebesgue measure $d\sigma_1$ and $d\sigma_2$ respectively. If $0 < q \leq \infty$, we say that the bilinear adjoint restriction estimate $R^*_{S_1,S_2}(2 \times 2 \to q)$ holds if one has

$$\|\prod_{t=1}^{2} \widehat{f_t d\sigma_t}\|_{L^q(\mathbf{R}^3)} \lesssim \prod_{t=1}^{2} \|f_t\|_2.$$

for all test functions f_1, f_2 supported on S_1, S_2 respectively.

The question of finding, for specified S_1, S_2, the range of q for which $R^*_{S_1,S_2}(2 \times 2 \to q)$ holds is open. The estimate is trivial for $q = \infty$, and if $S_1 = S_2$ is a subset of a hyperplane then no estimate is possible for $q < \infty$. On the other hand, if S_1 and S_2 are transverse in the sense that the normals to S_1 lie in a set which is separated from the normals to S_2, then we have the estimate for $q = 2$ (and hence for $q \geq 2$ by interpolation). Indeed, by Plancherel's theorem $R^*_{S_1,S_2}(2 \times 2 \to 2)$ is equivalent to

$$\|f_1 d\sigma_1 * f_2 d\sigma_2\|_2 \lesssim \|f_1\|_2 \|f_2\|_2,$$

and the result follows by interpolating between the trivial estimate

$$\|f_1 d\sigma_1 * f_2 d\sigma_2\|_1 \lesssim \|f_1\|_1 \|f_2\|_1,$$

and the estimate

$$\|f_1 d\sigma_1 * f_2 d\sigma_2\|_\infty \lesssim \|f_1\|_\infty \|f_2\|_\infty$$

which follows from transversality. If S_1, S_2 are not transverse but have non-vanishing Gaussian curvature then one can also obtain the estimate $R^*_{S_1,S_2}(2 \times 2 \to 2)$ from the Tomas-Stein theorem and Hölder's inequality.

If one assumes only that S_1 and S_2 are transverse, then the above argument is sharp, as can be seen by taking S_1 and S_2 to be subsets of transverse hyperplanes. The argument is still sharp for certain curved sets S_1, S_2; for instance if S_1 and S_2 are unit-seperated subsets of the cylinder

$$\{(\xi_1, \xi_2, \xi_3) : |\xi_1|^2 + |\xi_2|^2 = 1\},$$

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then by taking f_1, f_2 to be bump functions one can make $\widehat{f_t d\sigma_t}$ comparable to $R^{-1/2}$ on (disjoint) $R \times R \times 1$ slabs for some arbitrarily large R, as a simple stationary phase computation shows. By multiplying f_1 or f_2 by a suitable phase one can make the two slabs co-incide, and so the estimate $R^*_{S_1,S_2}(2 \times 2 \to q)$ is impossible for $q < 2$.

However, it was observed by Bourgain [3] that the estimate $R^*_{S_1,S_2}(2 \times 2 \to q)$ can hold for some $q < 2$ if the surfaces S_1 and S_2 have slightly more curvature than the cylinder example; in particular, if S_1 and S_2 are separated subsets of a small portion of the light cone then $R^*_{S_1,S_2}(2 \times 2 \to 2 - \tau)$ holds for some small $\tau > 0$. Indeed, examples suggest that one should have $R^*_{S_1,S_2}(2 \times 2 \to q)$ for all $q \geq 5/3$ in this case (see [34, 35]; this conjecture is due to Machedon and Klainerman).

Estimates of this form were studied in [35], the prequel to this paper. We state the two relevant results of that paper here. The first result pertains to pairs of surfaces which resemble projectively separated subsets of the light cone, or of transverse subsets of non-parallel cylinders.

Definition 1.1. Suppose that S_1 and S_2 are compact surfaces with boundary in $\mathbf{R}^3$. If $\xi \in S_t$, $t = 1, 2$, we use $n_t(\xi) \in S^2/\pm$ to denote the unit normal to S_t at ξ. We say that the pair S_1 and S_2 are of *disjoint conic type* if the following statements hold:

- (Transversality) For all $\xi_t \in S_t$, $t = 1, 2$ we have $n_t(\xi_t) \in N_t$, where N_1 and N_2 are small disjoint caps in $S^2/\pm$ which are seperated by a distance comparable to 1.
- (Null direction) The map $dn_t : T_{\xi_t} S_t \to T_{\xi_t} S_t$ has eigenvalue 0 with multiplicity one in the direction $w_t(\xi_t) \in S^2/\pm$. We also assume that the remaining eigenvalue has magnitude ~ 1.
- (Tranversality of null directions) For all $\xi_t \in S_t$, $t = 1, 2$ we have $w_t(\xi_t) \in W_t$, where W_1 and W_2 are small disjoint caps in $S^2/\pm$ which are seperated by a distance comparable to 1. Furthermore, the maximal angular seperation between W_1 and N_2, or W_2 and N_1, is strictly less than $\pi/2$.

Theorem 1.2. [35] *If S_1 and S_2 are of disjoint conic type, then $R^*_{S_1,S_2}(2 \times 2 \to q)$ for all $q > 2 - \frac{8}{121}$.*

If one assumes more curvature on S_1, S_2 (for instance of S_1, S_2 are separated subsets of the sphere or paraboloid) then better results are known. More precisely:

Theorem 1.3. [35] *Let S be a graph of an elli ptic phase function (in the sense of* [34] *or* [18, 19]*), and let S_1, S_2 be unit-seperated subsets of S. Then*

$$R^*_{S_1,S_2}(2 \times 2 \to q) \tag{1}$$

for all $q > 2 - \frac{2}{17}$.

The purpose of this paper is to give applications of these results to outstanding conjectures in harmonic analysis and wave equations. We will use the philosophy of [34] to allow us to pass efficiently from the bilinear estimates stated above to the linear estimates which appear in applications.

We have the following results. Firstly, in Section 2 we show pointwise convergence of solutions of the free Schrödinger equation to their initial data for low regularities. More precisely, we have pointwise convergence for all data in H^s, $s > 15/32$. This improves on previous work by [2], [18].

In Section 3 we show that Theorem 1.2 implies some new null form estimates for the $L^q_{x,t}$ norms of products of solutions to the wave equation in $\mathbf{R}^{2+1}$. This gives some progress on a recent conjecture of Foschiand Klainerman [12]. As an application of these null form estimates we improve upon the local smoothing estimate proven by Schlag and Sogge [28].

In [3], the estimates of the type considered in Theorem 1.2 were used to obtain an improvement to Mockenhaupt's square function estimate [20]. By combining this argument with the bilinear philosophy in [34] we are able to obtain a quantitative improvement, reducing the $N^{1/4}$ factor in Mockenhaupt's estimate to $1/4 - 1/238$. We also prove the corresponding amount of progress on the Bochner-Riesz multiplier problem, Sogge's local smoothing conjecture, and the L^p smoothing conjecture for the convolution operator with the helix.

FIGURE 1. Logical layout of this paper.

Although there are many common themes in these results, particularly the bilinear philosophy and the use of the estimates $R^*(2 \times 2 \to q)$, some of the sections can be read independently, although all sections utilize some basic lemmas in harmonic analysis which we place in an Appendix. We display the logical dependencies of the results of this paper in Figure 1.

2. Application to the Schrodinger maximal function

Consider the solution $e^{it\Delta}f$ of the free Schrödinger equation with initial datum f. We are interested in proving a.e. pointwise convergence $e^{it\Delta}f \to f(x)$ when $t \to 0$, under certain assumptions on the (weak) regularity of f. We follow the line started by Carleson [5] who proved convergence for f in the Sobolev space $H^{1/4}(\mathbf{R})$. This is a one dimensional result. For $n \geq 3$, the best result known was proven independently by Sjölin [29] and Vega [36], who showed convergence for $f \in H^s(\mathbf{R}^n)$ whenever $s > 1/2$. For functions f in $\mathbf{R}^2$, due to a better knoweledge of the restriction properties of the paraboloid in $\mathbf{R}^3$, convergence has been proven under weaker assumptions. Bourgain [2], [4] proved that there is $s < 1/2$ for which the result holds. This theorem was refined by Moyua, Vargas and Vega [18], and the argument there could be improved using the results of Tao, Vargas and Vega in [34] and those in [35]. However we provide a simpler argument in this section based on the bilinear philosophy which gives slightly better exponents.

Theorem 2.1. *Suppose that q is such that $R^*_{S_1,S_2}(2\times 2 \to q)$ whenever S_1, S_2 are unit-separated subsets of the standard paraboloid in $\boldsymbol{R}^3$. Then, for all $s > 1 - 1/q$ and all $f \in H^s(\boldsymbol{R}^2)$,*

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \qquad a.e.\ x \in \boldsymbol{R}^2.$$

Combining this with (1), we thus obtain pointwise convergence of solutions to the free Schrödinger equation to their initial data, if the data is in H^s for $s > 15/32 = .46875$. This improves upon the results in [18], which required

$$s > \frac{164+\sqrt{2}}{339} = .4879475\ldots$$

As usual, the convergence will be a consequence of an estimate for a maximal function.

Theorem 2.2. *Under the same assumptions*[1],

$$\|\sup_t \Big| \int \widehat{f}(\xi) e^{2\pi i(<x,\xi>+t|\xi|^2)}\, d\xi \Big| \|_{L^{2q}(\boldsymbol{R}^2)} \lesssim \|f\|_{H^s(\boldsymbol{R}^2)}.$$

Note that this estimate is somewhat stronger than what is needed for pointwise convergence. By the usual arguments of Littlewood-Paley decomposition and Plancherel's theorem, it suffices to prove the following proposition.

Proposition 2.3. *Under the same assumptions,*

$$\|\sup_t \Big| \int \widehat{f}(\xi) e^{2\pi i(<x,\xi>+t|\xi|^2)}\, d\xi \Big| \|_{2q} \leq N^{1-1/q} \|f\|_2. \tag{2}$$

for all $N > 0$ and all f with supp $f \subset B(0,N)\backslash B(0,N/2)$,

[1] All estimates in this paper will be a priori.

Proof

We rescale to make the support of our function independent of N:

$$\|\sup_t \Big| \int \widehat{f}(\xi) e^{2\pi i(<x,\xi>+t|\xi|^2)}\, d\xi \Big| \|_{2q}$$
$$= N^{2-1/q} \|\sup_t \Big| \int_{B(0,1)} \widehat{f}(N\xi) e^{2\pi i(<x,\xi>+t|\xi|^2)}\, d\xi \Big| \|_{2q}.$$

By Plancherel's theorem, it thus suffices to show that we havE

$$\|\sup_t |\widehat{g\, d\sigma}|\|_{2q} \le C\|g\|_2$$

for all functions g supported in a compact subset S of the paraboloid. We square this as

$$\|\sup_t |\widehat{g\, d\sigma} \widehat{g\, d\sigma}|\|_q \le C\|g\|_2\|g\|_2 \tag{3}$$

For each $j \ge 0$ we break up S into dyadic "squares" τ_k^j, and write $\tau_k^j \sim \tau_{k'}^j$ if $\tau_k^j, \tau_{k'}^j$ are not adjacent but have adjacent parents. For each $j \ge 0$ write $g = \sum g_k^j$ where $g_k^j(\xi) = g\chi_{\tau_k^j}(\xi)$. Then,

$$\widehat{g\, d\sigma}\widehat{g\, d\sigma} = \sum_j \sum_{k,k':\tau_k^j \sim \tau_{k'}^j} \widehat{g_k^j\, d\sigma}\widehat{g_{k'}^j\, d\sigma}.$$

From the triangle inequality, the left hand side of (3) is thus majorized by

$$\sum_j \sum_{k,k':\tau_k^j \sim \tau_{k'}^j} \|\sup_t |\widehat{g_k^j\, d\sigma}(\cdot,t)\widehat{g_{k'}^j\, d\sigma}(\cdot,t)|\|_q. \tag{4}$$

We fix j. From the geometry of τ_k^j we see then that the temporal Fourier transform of $\widehat{g_k^j\, d\sigma}(x,\cdot)$ is supported in an interval of length 2^{-j}. The same statement holds for $\widehat{g_k^j\, d\sigma}(x,\cdot)\widehat{g_{k'}^j\, d\sigma}(x,\cdot)$. Hence, we can use Lemma 7.3 to bound

$$\sup_t |\widehat{g_k^j\, d\sigma}(x,t)\widehat{g_{k'}^j\, d\sigma}(x,t)| \lesssim (2^{-j})^{1/q} \|\widehat{g_k^j\, d\sigma}(x,\cdot)\widehat{g_{k'}^j\, d\sigma}(x,\cdot)\|_{L^q(dt)}.$$

Therefore, we estimate (4) by

$$(\sum_j \sum_{k,k':\tau_k^j \sim \tau_{k'}^j} 2^{-j/q} \|\widehat{g_k^j\, d\sigma}\widehat{g_{k'}^j\, d\sigma}\|_{L^q(dtdx)})^{1/2}.$$

By $R^*_{S_1,S_2}(2 \times 2 \to q)$, an affine transformation, and parabolic rescaling (cf. [34], Proposition 2.6) the last expression is not bigger than

$$C(\sum_j \sum_{k,k':\tau_k^j \sim \tau_{k'}^j} 2^{-j/q} 2^{-2j} 2^{4j/q} \|g_k^j\|_2 \|g_{k'}^j\|_2)^{1/2}.$$

Finally, we sum on k and j and take square root. Note that $5/3 < q$ implies $3/q - 2 < 0$. ∎

The best estimate of the form $R^*(2 \times 2 \to q)$ that one can hope for is $q = 5/3$. Thus this argument can at best give convergence in H^s for $s > 2/5$. However, the strongest counterexample known to date (see [10]) only gives $s \geq 1/4$ as a necessary condition, which suggests that the above argument is not sharp. The use of Lemma 7.3 in the above seems particularly weak.

We remark that the above argument extends without difficulty to the more general dispersive equations studied in [19].

3. Null form estimates for L^p, $p < 2$

In this section we consider estimates on one-sided solutions $\phi^\pm = e^{\pm i\sqrt{-\Delta}t} f$ to the free wave equation in $\mathbf{R}^{2+1}$, in terms of Sobolev norms of the initial data f. (Estimates for general solutions to the free wave equation can be obtained by the usual decomposition $\phi = \phi^+ + \phi^-$ into one-sided solutions).

Strichartz' estimate [31] in $2+1$ dimensions states that

$$\|\phi^\pm\|_{L^6_{x,t}} \lesssim \|f\|_{\dot H^{1/2}}.$$

We may bilinearize this in the usual manner as

$$\|\phi^\pm \psi^\pm\|_{L^3_{x,t}} \lesssim \|f\|_{\dot H^{1/2}} \|g\|_{\dot H^{1/2}},$$

where $\phi^\pm = e^{\pm i\sqrt{-\Delta}t} f, \psi^\pm = e^{\pm i\sqrt{-\Delta}t} g$, and the two signs $\pm$ need not agree.

Without any multiplier weights on the quantity $\phi^\pm \psi^\pm$, this is the best estimate available in pure Lebesgue norms $L^q_{x,t}$. (If one allows mixed norms such as $L^q_t L^r_x$ then other estimates are possible; see e.g. [13], [12] and the references therein.)

However, if one allows weights on the right-hand side then there are many further estimates available. More precisely, if we let D_0, D_+ and D_- be the Fourier multipliers corresponding to the functions[2] $|\xi|$, $|\xi| + |\tau|$, $||\xi| - |\tau||$ respectively, then we may consider estimates of the form

$$\|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} (\phi^\pm \psi^\pm)\|_{L^q_{x,t}} \lesssim \|f\|_{\dot H^{\alpha_1}} \|g\|_{\dot H^{\alpha_2}} \tag{5}$$

for various real numbers $q, \beta_0, \beta_+, \beta_-, \alpha_1, \alpha_2$. In the remainder of the section we will always assume that $\alpha_1 \geq \alpha_2$, and that the sign on ϕ is positive; the remaining cases will follow from symmetry. We will distinguish two cases $(++)$ and $(+-)$ of (5), depending on the sign on $\psi^\pm$. Henceforth we fix the sign $\pm$ to be the same at every occurence.

When $q = 2$ such estimates are important in the study of semi-linear wave equations with quadratic non-linearities; such equations arise naturally in the study of wave maps. (See [refs?]) Estimates of this type with $q \neq 2$, $\beta_+ = \beta_- = 0$ and β negative have also been recently utilized in [17]. However for $q < 2$ no estimates were known in $\mathbf{R}^{2+1}$.

The study of these estimates and generalizations was systematically taken up in [12] (see also [14, 15, 16], and earlier work by [1]), and the following necessary conditions were found:

[2]In this section and the next, we use the variables $(\xi, \tau) = (\xi_1, \xi_2, \tau)$ to parameterize frequency space $\mathbf{R}^{2+1}$.

Proposition 3.1. [12] *With the above notation and assumptions, the inequalities*

$$2 - \frac{3}{q} + \beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 \tag{6}$$

$$q \geq \frac{5}{3}$$

$$\beta_- \geq \frac{3}{2q} - \frac{1}{2} \tag{7}$$

$$\alpha_1 \leq \beta_- + 2 - \frac{3}{q} \tag{8}$$

$$\alpha_1 \leq \beta_- + \frac{5}{4} - \frac{3}{2q}$$

$$\alpha_1 \leq \beta_- + 1$$

are necessary in order for (5) *to hold. In the* $(++)$ *case we have the additional necessary conditions*

$$\beta_0 \geq \frac{3}{q} - 2$$

$$\beta_0 \geq \frac{5}{q} - 3 \tag{9}$$

and in the $(+-)$ *case we have the additional necessary condition*

$$\alpha_1 + \alpha_2 \geq \frac{5}{q} - 2. \tag{10}$$

The condition (6) comes from scaling considerations, while the remaining conditions come from considering examples of f_1, f_2 with various frequency supports. Condition (7) is also related to Lorentz invariance considerations (see [12]). It was conjectured in [12] that (5) is true whenever all of the above necessary conditions hold, unless at least two of the inequalities hold with strict equality. This conjecture has been verified for $q = 2$ but is largely open otherwise.

We will concern ourselves with the case $\frac{5}{3} \leq q < 2$, for which no estimates of the type (5) were previously known to exist in $2+1$ dimensions. In this case many of the above conditions are redundant, and one only requires (6), (7), (8), (9) and (10).

Our results are as follows.

Theorem 3.2. *Let* $\frac{5}{3} \leq q \leq 2$ *be suchthat* $R^*_{S_1,S_2}(2 \times 2 \to q)$ *holds for all pairs* S_1, S_2 *of surfaces of disjoint conic type. Suppose that* (6) *holds,* (7) *is satisfied with strict inequality, and*

$$\alpha_1 < \beta_- + 2 - \frac{3}{q} - 4(\frac{1}{q} - \frac{1}{2}). \tag{11}$$

If

$$\alpha_1 + \alpha_2 > \frac{5}{q} - 2 + 4(\frac{1}{q} - \frac{1}{2}) \tag{12}$$

then the $(+-)$ *form of* (5) *holds. If*

$$\beta_0 > \frac{5}{q} - 3 + 2(\frac{1}{q} - \frac{1}{2}) \tag{13}$$

then the $(++)$ *form of* (5) *holds.*

In particular, by Theorem 1.2 the above result holds for all $2 - \frac{8}{121} < q \leq 2$. The factors of $(\frac{1}{q} - \frac{1}{2})$ represent some inefficiencies in our arguments; however in the endpoint case $q = \frac{5}{3}$ we have been able to replace (11), (13) by their sharp counterparts (8), (9) (except for endpoints).

Analogues of the above theorem also hold for the null forms

$$Q_0(\phi^+, \psi^\pm) = \partial_t \phi^+ \partial_t \psi^\pm + \nabla_x \phi^+ \cdot \nabla_x \psi^\pm$$
$$Q_{ij}(\phi^+, \psi^\pm) = \partial_{x_i} \phi^+ \partial_{x_j} \psi^\pm - \partial_{x_j} \phi^+ \partial_{x_i} \psi^\pm$$
$$Q_{0j}(\phi^+, \psi^\pm) = \partial_t \phi^+ \partial_{x_j} \psi^\pm - \partial_{x_j} \phi^+ \partial_t \psi^\pm;$$

From an analysis of the multiplier (see e.g. [12]), one expects these null-forms to behave like

$$Q_0(\phi^+, \psi^-) \sim D_+ D_- (\phi^+ \psi^-)$$
$$Q_{0j}(\phi^+, \psi^-) \sim D_+^{1/2} D_-^{1/2} (D_0^{1/2} \phi^- D_0^{1/2} \psi^-)$$
$$Q_{ij}(\phi^+, \psi^-) \sim D_+^{1/2} D_-^{1/2} (D_0^{1/2} \phi^+ D_0^{1/2} \psi^-)$$

in the $(+-)$ case and

$$Q_0(\phi^+, \psi^+) \sim D_+ D_- (\phi^+ \psi^+)$$
$$Q_{0j}(\phi^+, \psi^+) \sim D_+^{1/2} D_-^{1/2} (D_0^{1/2} \phi^+ D_0^{1/2} \psi^+)$$
$$Q_{ij}(\phi^+, \psi^+) \sim D_0 D_+^{-1/2} D_-^{1/2} (D_0^{1/2} \phi^+ D_0^{1/2} \psi^+)$$

in the $(++)$ case. A modification of the arguments below show that our results extend to these null forms as if the above heuristics were exact.

3.3. Preliminaries: Some Minkowski geometry. In this section we briefly state some elementary facts of Minkowski geometry that we shall need, together with some notation.

If Q is a subset of $\mathbf{R}^2$, we define the lifts $S^+(Q)$, $S^-(Q)$ of Q to the upper and lower light-cones respectively by

$$S^\pm(Q) = \{(\xi, \pm|\xi|) : \xi \in Q\}.$$

We give these light cones two measures: the induced measure $d\xi$ which is the push-forward of Lebesgue measure on $\mathbf{R}^2$ under $S^\pm$, and the invariant measure $\frac{d\xi}{|\xi|}$, which is left unchanged by the action of the unimodular Lorentz group $SO(2,1)$. Note that $d\xi$ is also comparable to the measure induced from the Euclidean metric on R^{2+1}.

The group generated by the Lorentz group, the isotropic dilations, is the group of orientation-preserving conformal linear transformations $Conf^+(\mathbf{R}^{2+1})$ of Minkowski space. This group has the following 2-transitivity property: if $A, A' \in S^+(\mathbf{R}^2), B, B' \in S^\pm(\mathbf{R}^2)$ are such that A, B are linearly independent and A', B' are linearly independent, then there exists a conformal linear transformation L such that $L(A) = A'$ and $L(B) = B'$. This fact will often allow us to place the Fourier transforms of f and g in specified locations in frequency space.

In order to handle the D_- weight, we will need the identity

$$|\xi+\eta| - ||\xi| - |\eta|| = \frac{|\xi+\eta|^2 - (|\xi| - |\eta|)^2}{|\xi+\eta| + ||\xi| - |\eta||} \sim \frac{|\xi||\eta|\angle(\xi,-\eta)^2}{|\xi+\eta|} \tag{14}$$

for the $(+-)$ case,and

$$|\xi| + |\eta| - |\xi+\eta| = \frac{(|\xi| + |\eta|)^2 - |\xi+\eta|^2}{|\xi| + |\eta| + |\xi+\eta|} \sim \frac{|\xi||\eta|\angle(\xi,\eta)^2}{|\xi| + |\eta|} \tag{15}$$

for the $(++)$ case; here $\angle(\xi, \eta)$ denotes the angle subtended by ξ and η with respect to the origin.

At certain stages in the argument it will be convenient to view the light cone in null co-ordinates (a_1, a_2, a_3) defined by

$$(a_1, a_2, a_3) = (\frac{\tau+\xi_1}{2}, \xi_2, \frac{\tau-\xi_1}{2}).$$

The double light cone $|\tau| = |\xi|$ in these co-ordinates becomes

$$4a_1a_3 = a_2^2 \tag{16}$$

with the upper and lower light cone corresponding to $a_1, a_3 \geq 0$ and $a_1, a_3 \leq 0$ respectively. Note that for any constants $A, B, C > 0$ such that $AC = B^2$, the map

$$(a_1, a_2, a_3) \to (Aa_1, Ba_2, Ca_3)$$

preserves the light cone and is hence a conformal linear transformation. If two sets Q^1, Q^2 are very close in one of the co-ordinates (especially the a_2 co-ordinate), we will frequently apply the above type of transformation to make the sets Q^1, Q^2 unit-separated in this co-ordinate.

3.4. Preliminaries: The quantity $C^\pm$. In this section we recast the desired estimate (5) in terms of a quantity $C^\pm_{\alpha_1,\alpha_2}(Q^1, Q^2, D)$, which we will shortly define. We then collect some basic properties of $C^\pm$ to use in the proof of Theorem 3.2. Henceforth $q < 2$ will be fixed to be such that $R^*_{S_1,S_2}(2 \times 2 \to q)$ for all surfaces S_1, S_2 of disjoint conic type.

Definition 3.5. If $\alpha_1, \alpha_2 \in \mathbf{R}$, Q^1, Q^2 are subsets of $\mathbf{R}^2$ and D is a Fourier multiplier on $\mathbf{R}^{2+1}$ given by $m(\xi, \tau)$, we define $C^\pm_{\alpha_1,\alpha_2}(Q^1, Q^2, D)$ to be the best constant C (possibly infinite) such that

$$\|D(\phi^+\psi^\pm)\|_{L^q_{x,t}} \leq C\|f\|_{\dot H^{\alpha_1}}\|g\|_{\dot H^{\alpha_2}}$$

for all f, g whose Fourier transforms are supported in Q^1, Q^2 respectively.

Thus (5) can be written as

$$C^{\pm}_{\alpha_1,\alpha_2}(\mathbf{R}^2, \mathbf{R}^2, D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}) \lesssim 1. \tag{17}$$

On the other hand, from Plancherel's theorem we see that $C^{\pm}_{0,0}(Q^1, Q^2, 1)$ is essentially the best constant C that appears in the estimate

$$\|\widehat{fd\sigma_1}\widehat{gd\sigma_2}\|_q \leq C\|f\|_2\|g\|_2 \tag{18}$$

for all f and g supported on the surfaces $S^+(Q^1)$, $S^{\pm}(Q^2)$ respectively, where $d\sigma_1$, $d\sigma_2$ are induced measure $d\xi$ as defined in Section 3.3. From the hypothesis $R^*(2 \times 2 \to q)$ we thus have

Proposition 3.6. *If Q^1, Q^2 are such that the surfaces $S^+(Q^1)$, $S^{\pm}(Q^2)$ are of disjoint conic type, then*

$$C^{\pm}_{0,0}(Q^1, Q^2, 1) \lesssim 1$$

where the constant depends only on the constants in the disjoint conic type condition.

Our strategy shall be to use the familiar techniques of dyadic decomposition and rescaling to reduce (5) to something treatable by Proposition 3.6. Due to the geometry of the cone some of our decompositions and rescalings will be non-isotropic, and some of our techniques will not be totally efficient.

We now formalize abstractly the decomposition and rescaling properties that we shall need. We first observe the trivial relationship

$$C^{\pm}_{\alpha_1,\alpha_2}(Q^1, Q^2, D_1 + D_2) \leq C^{\pm}_{\alpha_1,\alpha_2}(Q^1, Q^2, D_1) + C^{\pm}_{\alpha_1,\alpha_2}(Q^1, Q^2, D_2). \tag{19}$$

Also, if P is a multiplier which is bounded on L^q (in particular, if P is in one of the standard multiplier classes) then we have

$$C^{\pm}_{\alpha_1,\alpha_2}(Q^1, Q^2, PD) \leq C^{\pm}_{\alpha_1,\alpha_2}(Q^1, Q^2, D). \tag{20}$$

We also observe a scaling property, which can be deduced from dimensional analysis.

Lemma 3.7. *If j is an integer, and the notation is as in Definition 3.5, then*

$$C^{\pm}_{\alpha_1,\alpha_2}(Q^1, Q^2, D) = 2^{(\frac{3}{q}+\alpha_1+\alpha_2-2)j} C^{\pm}_{\alpha_1,\alpha_2}(2^j Q^1, 2^j Q^2, D_{2^j}),$$

where D_j is given by the multiplier

$$m_{2^j}(\xi, \tau) = m(2^{-j}\xi, 2^{-j}\tau).$$

In the case when $\alpha_1 = \alpha_2 = 1/2$ there is an additional scaling because the $\dot{H}^{1/2}$ norm is invariant under Lorentz transformations of $\phi^+, \psi^{\pm}$. Indeed, from the identity

$$\widehat{\phi^+}(\xi, \tau) = C\hat{f}(\xi)\delta(\tau - |\xi|)\, d\xi,$$

where we view the spacetime Fourier transform of ϕ^+ as a measure, we can write the $\dot{H}^{1/2}$ norm of f in an invariant way as

$$\|f\|_{\dot{H}^{1/2}} = C\|\frac{\widehat{\phi^+}}{d\mu}\|_{L^2(d\mu)}$$

where $d\mu$ is the invariant measure $d\xi/|\xi|$ on the upper light cone and $\frac{\widehat{\phi^+}}{d\mu}$ denotes the Radon-Nikodym derivative. A similar argument works for $\psi^\pm$.

Since Lorentz transformations are unimodular, they preserve L^q and we obtain the following convenient fact.

Lemma 3.8. *If $L \in SO(2,1)$ is a Lorentz transformation, Q^1, Q^2 are subsets of $\boldsymbol{R}^2$ and D is a Fourier multiplier given by $m(\xi,\tau)$, then*

$$C^{\pm}_{1/2,1/2}(Q^1, Q^2, D) = C^{\pm}_{1/2,1/2}(Q^1_L, Q^2_L, D_L),$$

where Q^1_L, Q^2_L are given by

$$S^+(Q^1_L) = L(S^+(Q^1)), \quad S^\pm(Q^2_L) = L(S^\pm(Q^2)),$$

and D_L is given by the multiplier $m \circ L^{-1}$.

Combining these two lemmas we have

Corollary 3.9. *If L is an element of $Conf^+(\boldsymbol{R}^{2+1})$, Q^1, Q^2 are subsets of $\boldsymbol{R}^2$ and D is a Fourier multiplier given by $m(\xi,\tau)$, then*

$$C^{\pm}_{1/2,1/2}(Q^1, Q^2, D) = |\det L|^{\frac{1}{q}-\frac{1}{3}} C^{\pm}_{1/2,1/2}(Q^1_L, Q^2_L, D_L),$$

where Q^1_L, Q^2_L, D_L are as in Lemma 3.8.

We now discuss the problem of controlling $C^\pm(Q^1, Q^2, D)$ by a suitable partition of Q^1, Q^2. We begin with a variant of Schur's test.

Lemma 3.10. *Suppose the notation is as in Definition 3.5, and suppose that we are given partitions $Q^1 = \bigcup_k Q^1_k$, $Q^2 = \bigcup_l Q^2_l$ into essentially disjoint sets. Suppose that $A > 0$ is such that*

$$\sup_k \sum_l C^{\pm}_{\alpha_1,\alpha_2}(Q^1_k, Q^2_l, D) < A$$

and

$$\sup_l \sum_k C^{\pm}_{\alpha_1,\alpha_2}(Q^1_k, Q^2_l, D) < A.$$

Then

$$C^{\pm}_{\alpha_1,\alpha_2}(Q^1, Q^2, D) \lesssim A.$$

Proof Fix f, g with Fourier transforms supported on Q^1, Q^2 respectively, and decompose $f = \sum_k f_k$, $g = \sum_l g_l$, $\phi^+ = \sum_k \phi^+_k$, $\psi^+ = \sum_l \psi^+_l$ accordingly. We need to estimate

$$\| \sum_k \sum_l D(\phi^+_k \psi^+_l) \|_q. \tag{21}$$

By the triangle inequality and Definition 3.5, this is majorized by

$$\sum_{k,l} C^{\pm}_{\alpha_1,\alpha_2}(Q^1_k, Q^2_l, D) \|f_k\|_{\dot H^{\alpha_1}} \|g_l\|_{\dot H^{\alpha_2}}$$

But by the hypothesis on k, l, and Schur's test, we may majorize this by

$$A(\sum_k \|f_k\|^2_{\dot H^{\alpha_1}})^{1/2} (\sum_l \|g_l\|^2_{\dot H^{\alpha_2}})^{1/2}$$

and the claim follows from Definition 3.5 and Plancherel's theorem. ■

With the notation as above, we observe that the space-time Fourier transform of $D(\phi^+\psi^\pm)$ is supported on the set

(22)
$$supp(m) \cap (S^+(Q^1) + S^\pm(Q^2)) = supp(m) \cap \{(\xi+\eta, |\xi| \pm |\eta|) : \xi \in Q^1, \eta \in Q^2\}.$$

Thus, if this set is empty, then $C^\pm_{\alpha_1,\alpha_2}(Q^1, Q^2, D) = 0$. This gives an immediate corollary of Lemma 3.10:

Corollary 3.11. *Suppose the notation is as in Lemma* 3.10. *Write* $Q^1_k \sim Q^2_l$ *if there exists* $\xi \in Q^1_k$, $\eta \in Q^2_l$ *such that*

$$m(\xi+\eta, |\xi| \pm |\eta|) \neq 0.$$

Suppose further that for each k *(resp.* l*) there are at most* $O(1)$ l *(resp.* k*) such that* $Q^1_k \sim Q^2_l$. *Then*

$$C^\pm_{\alpha_1,\alpha_2}(Q^1, Q^2, D) \lesssim \sup_{k,l:Q^1_k \sim Q^2_l} C^\pm_{\alpha_1,\alpha_2}(Q^1_k, Q^2_l, D).$$

Corollary 3.11 is quite efficient, but it only works when the relation $Q^1_k \sim Q^2_l$ is only sparsely satisfied. In the general case we have Lemma 3.10, but this gives poor estimates. We can improve on this Lemma by exploiting quasi-orthogonality, although we have not been able to obtain the optimal exponents in this manner. More precisely, we have

Lemma 3.12. *Suppose the notation is as in Lemma* 3.10. *Suppose also that for distinct* (k, l) *the sets*

$$S^+(Q^1_k) + S^\pm(Q^2_l)$$

lie in essentially disjoint rectangles in R^{2+1}. *Then we have*

$$C^\pm_{\alpha_1,\alpha_2}(Q^1, Q^2, D) \lesssim (KL)^{\frac{1}{q}-\frac{1}{2}} \sup_{k,l} C^\pm_{\alpha_1,\alpha_2}(Q^1_k, Q^2_l, D)$$

where K, L *are the cardinalities of the index sets of* k, l *respectively.*

Proof We repeat the proof of Lemma 3.10. This time, however, we exploit the hypothesis, which implies by (22) that the functions $D(\phi^+_k\psi^+_l)$ have Fourier transforms supported in essentially disjoint rectangles. Thus we use Lemma 7.1 instead of the triangle inequality, to majorize (21) by

$$(\sum_k \sum_l \|D(\phi^+_k\psi^+_l)\|^q_q)^{1/q}.$$

By Hölder's inequality this is majorized by

$$(KL)^{\frac{1}{q}-\frac{1}{2}} (\sum_k \sum_l \|D(\phi^+_k\psi^+_l)\|^2_q)^{1/2}.$$

By Definition 3.5 this is majorized by

$$(KL)^{\frac{1}{2}-\frac{1}{q}} \sup_{k,l} C^\pm_{\alpha_1,\alpha_2}(Q^1_k, Q^2_l, D) (\sum_k \sum_l \|f_k\|^2_{\dot H^{\alpha_1}} \|g_l\|^2_{\dot H^{\alpha_2}})^{1/2}.$$

The claim then follows from Definition 3.5 and Plancherel's theorem. ■

To close this section we give an easy consequence of Plancherel's theorem.

Lemma 3.13. *Let the notation be as in Definition* 3.5. *If* Q^1, Q^2 *are supported on the annuli* $\{\xi : |\xi| \sim 2^j\}$, $\{\eta : |\eta| \sim 2^k\}$, *then*

$$C^{\pm}_{\alpha_1,\alpha_2}(Q^1, Q^2, D) \sim 2^{-\alpha_1 j} 2^{-\alpha_2 k} C^{\pm}_{0,0}(Q^1, Q^2, D).$$

3.14. Proof of Theorem 3.2. The first part of the proof consists of repeated applications of the above decomposition and rescaling lemmas until all the multipliers are essentially constant, and we are in the one of three model cases for some $j \geq 0$:

FIGURE 2. Case I_j, viewed from the ξ_2 and τ axes. The supports of $\hat{\phi}$ and $\hat{\psi}$ are on almost diametrically opposite sides of the same light cone, so that $\widehat{\phi\psi}$ is close to the τ axis.

- Case I_j: We are in the $(++)$ case, and $\hat{f}$, $\hat{g}$ are supported on the sets

  (23)
  $$Q^1_{I_j} = \{\xi_1 = 1 + O(2^{-j}), \xi_2 \sim 2^{-j}\}, \quad Q^2_{I_j} = \{\eta_1 = -1 + O(2^{-j}), \eta_2 \sim 2^{-j}\}$$

  respectively. In this case D_0, D_+, D_- are comparable to 2^{-j}, 1, and 1 respectively. This case is good when β_0 is sufficiently large.
- Case II_j: We are in the $(+-)$ case, and $\hat{f}$, $\hat{g}$ are supported on the sets

  (24)
  $$Q^1_{II_j} = \{\xi : \xi_1 = 1 + O(2^{-j}), \xi_2 \sim 2^{-j}\}, \quad Q^2_{II_j} = \{\eta : \eta_1 = -1 + O(2^{-j}), \eta_2 \sim 2^{-j}\}$$

  respectively. In this case D_0, D_+, D_- are comparable to 2^{-j}, 2^{-j}, and 2^{-2j} respectively. This case is good when $\alpha_1 - \beta_-$ is sufficiently small.

- Case III_j: We are in either the $(++)$ or $(+-)$ case, and and $\hat{f}$, $\hat{g}$ are supported on the sets

$$(25)\, Q^1_{III_j} = \{\xi : \xi_1 \sim 1, |\xi_2| \ll 1\}, \quad Q^2_{III_j} = \{\eta : \eta_1 \sim \mp 2^j, |\eta_2| \ll 2^j\}$$

respectively. In this case D_0, D_+, D_- are comparable to 2^j, 2^j, and 1 respectively. This case is good when $\alpha_1 + \alpha_2$ is sufficiently large.

The above estimates on D_- may be verified by (14), (15). We illustrate the frequency supports of ϕ, ψ, and $\phi\psi$ schematically in Figures 2, 3, 4. Note that the perspectives are different in each case.

FIGURE 3. Case II_j, viewed from the ξ_1 and τ axes. The supports of $\hat{\phi}$ and $\hat{\psi}$ are on almost diametrically opposite sides of opposing light cones, so that $\widehat{\phi\psi}$ is close to the origin.

Our techniques require breaking the functions f_1, f_2 into many dyadic pieces, and summing them in fairly crude ways; this causes a logarithmic loss in many of our estimates. However, since we are assuming that the inequalities (7), (11), (12), (13) are satisfied with strict inequality, this loss is of no importance. It is quite likely that a more careful treatment would allow one to recover many of the endpoints. We remark that these reductions are valid for all q and all dimensions n (with (7) generalized to $\beta_- \geq \frac{n+1}{2q} - \frac{n-1}{2}$).

We then treat each of the model cases using further decompositions, Lemma 3.12, rescaling, and the hypothesis $R^*(2 \times 2 \to q)$. Our techniques here are not optimal. One possibility for improvement would be to use Littlewood-Paley theory for equally spaced projections in frequency space (cf. Lemma 7.2) and develop vector-valued analogues of the estimate $R^*(2 \times 2 \to q)$.

We begin the reduction to these cases. By Definition 3.5 it suffices to prove the estimate (17). We partition $\mathbf{R}^2$ into the dyadic annuli $A_k = \{\xi : |\xi| \sim 2^k\}$ for

FIGURE 4. Case III_j in the $(++)$ case, viewed in cross-section with the ξ_1, τ plane and from the τ axis. The supports of $\hat{\phi}$ and $\hat{\psi}$ are at widely separated angles and scales.

integer k. By Lemma 3.10, (17) will follow from the estimate

$$C^{\pm}_{\alpha_1,\alpha_2}(A_m, A_k, D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}) \lesssim 2^{-\varepsilon|m-k|}$$

uniformly in m, k for some $\varepsilon > 0$. From the assumption $\alpha_1 > \alpha_2$, symmetry, and Lemma 3.13 it suffices to verify the case when $k \geq m$. Accordingly we write $k = m + j$ for some $j \geq 0$. From Lemma 3.7 and the scaling condition (6) we may take m to equal 0, so we need only prove

$$C^{\pm}_{\alpha_1,\alpha_2}(A_0, A_j, D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}) \lesssim 2^{-\varepsilon j}.$$

We now split the multiplier $D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}$ into several pieces. In the $(++)$ case we decompose the identity as $I = P_+ + (I - P_+)$, where P_+, $(I - P_+)$ are given by multipliers supported on $|\xi| \ll |\tau|, |\xi| \gtrsim |\tau|$ respectively. In the $(+-)$ case we decompose the identity as $I = P_- + (I - P_-)$, where P_-, $(I - P_-)$ are given by multipliers supported on $|\xi| + |\tau| \ll 2^j, |\xi| + |\tau| \sim 2^j$ respectively.

By (19), we reduce to showing that

$$C^{+}_{\alpha_1,\alpha_2}(A_0, A_j, P_+ D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}) \lesssim 2^{-\varepsilon j}. \tag{26}$$

$$C^{-}_{\alpha_1,\alpha_2}(A_0, A_j, P_- D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}) \lesssim 2^{-\varepsilon j}. \tag{27}$$

$$C^{\pm}_{\alpha_1,\alpha_2}(A_0, A_j, (I - P_\pm) D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}) \lesssim 2^{-\varepsilon j}. \tag{28}$$

These three estimates will reduce to the cases I_j, II_j, III_j respectively (possibly after redefining j).

3.15. The contribution of P_+. The argument for this section is closely related to the one in [17].

Let f, g have Fourier transforms supported on A_0, A_j respectively. From (22) and the definition of P_+, we see that $P_+ D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi^+\psi^+)$ vanishes unless $j \leq C$. By enlarging the annuli A_k if necessary we may assume that $j = 0$. The indices α_1, α_2 are now irrelevant by Lemma 3.13, and will be discarded.

The region given by (22) is essentially a ball of unit radius, on which $D_-^{\beta_-}$ and $D_+^{\beta_+}$ are standard multipliers which can then be discarded by (20). Thus (26) reduces to

$$C^+(A_0, A_0, P_+ D_0^{\beta_0}) \lesssim 1.$$

We now perform another decomposition. For all $j \geq C$, let Δ_j^0 be given by a multiplier adapted to the cylindrical region $\{(\xi, \tau) : |\xi| \sim 2^{-j}\}$. Then on the region given by (22) we essentially have

$$P_+ D_0^{\beta_0} = \sum_{j \geq C} 2^{-\beta_0 j} \Delta_j^0,$$

so by (19) it suffices to show that

$$C^+(A_0, A_0, \Delta_j^0) \lesssim 2^{(\beta_0 - \varepsilon)j}.$$

Fix j, and divide A_0 into essentially disjoint squares Q of width 2^{-j}. With the notation of Corollary 3.11 with $D = \Delta_j^0$, we observe that $Q^1 \sim Q^2$ only holds when $Q^1 + Q^2 \subset B(0, C2^{-j})$. Thus the criteria of Corollary 3.11 are satisfied, and we reduce to showing that

$$C^+(Q^1, Q^2, \Delta_j^0) \lesssim 2^{(\beta_0 - \varepsilon)j}$$

uniformly for all Q^1, Q^2 such that $Q^1 + Q^2 \subset B(0, C2^{-j})$.

Fix Q^1, Q^2. By a rotation we may place these cubes on the ξ_1 axis, and by Lemma 3.7 we may make them a distance 1 fromthe origin, so that we are now in the case I_j mentioned at the start of this section. Discarding now the Δ_j^0 projection using (20), we will be done if we can show that

$$C^+(Q^1_{I_j}, Q^2_{I_j}, 1) \lesssim 2^{(\beta_0 - \varepsilon)j}. \tag{29}$$

Note that this is (except for the epsilon) the same estimate one would be faced with when specializing (5) to the case I_j. When j is small then this follows from Proposition 3.6 so we may assume that $j \gg 1$.

We would like to apply Proposition 3.6 for all j, but unfortunately the surfaces $S^+(Q^1_{I_j})$, $S^+(Q^2_{I_j})$ are too flat to be of disjoint conic type uniformly in j (after rescaling), and we must decompose and rescale further. Partition $Q^1_{I_j}$ (resp. $Q^2_{I_j}$) into about 2^j rectangles Q_1 (resp. Q_2) each of length 2^{-2j} in the e_1 direction and 2^{-j} in the e_2 direction. From a Taylor expansion of $|\xi|$ we see that the conditions of Lemma 3.12 are satisfied, so we have

$$C^+(Q^1_{I_j}, Q^2_{I_j}, 1) \lesssim 2^{2j(\frac{1}{q} - \frac{1}{2})} \sup_{Q^1, Q^2} C^+(Q^1, Q^2, 1). \tag{30}$$

By Corollary 3.9 we may assume that

$$Q^1 = \{\xi_1 = 1 + O(2^{-2j}), \xi_2 \sim 2^{-j}\}, \quad Q^2 = \{\eta_1 = -1 + O(2^{-2j}), \eta_2 \sim 2^{-j}\}.$$

By (18) we thus need to estimate the quantity

$$\|\widehat{fd\sigma_1}\widehat{gd\sigma_2}\|_q$$

for all f and g supported on $S^+(Q^1)$, $S^+(Q^2)$ respectively. In null co-ordinates these two surfaces can be written as

$$S^+(Q^1) = \{(a_1, a_2, a_3) : a_1 = 1 + O(2^{-2j}), a_2 \sim 2^{-j}, a_3 = \frac{a_2^2}{4a_1}\}$$

$$S^+(Q^2) = \{(a_1, a_2, a_3) : a_3 = 1 + O(2^{-2j}), a_2 \sim 2^{-j}, a_1 = \frac{a_2^2}{4a_3}\}$$

We now rescale these surfaces to be of unit size, applying the (non-conformal) scaling

$$L(a_1, a_2, a_3) = (2^{2j}a_1, 2^j a_2, 2^{2j}a_3).$$

The surfaces transform to

$$L(S^+(Q^1)) = \{(a_1, a_2, a_3) : a_1 = 2^{2j} + O(1), a_2 \sim 1, a_3 = \frac{a_2^2}{2^{-2j+2}a_1}\}$$

$$L(S^+(Q^2)) = \{(a_1, a_2, a_3) : a_3 = 2^{2j} + O(1), a_2 \sim 1, a_1 = \frac{a_2^2}{2^{-2j+2}a_3}\}.$$

Since $2^{-2j+2}a_1$, $2^{-2j+2}a_3$ are almost constant, these surfaces are essentially unit parabolic cylinders oriented in the $(1, 0, 0)$ and $(0, 0, 1)$ directions. These two surfaces are of disjoint conic type, and so the hypothesis $R^*(2 \times 2 \to q)$ applies. Undoing the scaling, we obtain

$$2^{-5j/q}\|\widehat{fd\sigma_1}\widehat{gd\sigma_1}\|_q \lesssim 2^{-3j/2}\|f\|_2 2^{-3j/2}\|g\|_2.$$

Thus

$$C^+(Q^1, Q^2, 1) \lesssim 2^{-3j}2^{5j/q}.$$

Combining this with (30) we obtain

$$C^+(Q^1_{I_j}, Q^2_{I_j}, 1) \lesssim 2^{2j(\frac{1}{q}-\frac{1}{2})}2^{-3j}2^{5j/q} \tag{31}$$

and (29) follows from (13).

For $q < 7/4$ there is a better way to prove (29), simply by replacing $Q^1_{I_j}$, $Q^2_{I_j}$ by the larger sets $Q^1_{I_0}$, $Q^2_{I_0}$ and using Proposition 18. This proves (29) for $\beta_0 > 0$ (and can also capture the endpoint $\beta_0 = 0$, simply by refusing to decompose using Δ_j). If we could prove the optimal estimate $R^*(2 \times 2 \to 5/3)$ then one would obtain the optimal value of β_0 given by (9).

3.16. The contribution of P_-. By arguing similarly to the previous section, we may assume that $j = 0$ and discard the α_1, α_2 subscripts.

On the region given by (22), $|\xi| + |\tau| \sim |\xi|$ and so $D_0^{\beta_0}$ is equal to $D_+^{\beta_0}$ times a harmless multiplier. Thus by (20) it suffices to show that

$$C^-(A_0, A_0, P_- D_+^{\beta_0+\beta_+} D_-^{\beta_-}) \lesssim 1.$$

On the region given by (22), we essentially have the decomposition

$$P_- D_+^{\beta_0+\beta_+} D_-^{\beta_-} = \sum_{j \geq C} \sum_{m \geq 0} 2^{-(\beta_0+\beta_+)j} 2^{-\beta_-(j+m)} \Delta_j^+ \Delta_{j+m}^-,$$

where Δ_j^+, Δ_j^- are given by multipliers which are bump functions adapted to $|\xi| + |\tau| \sim 2^{-j}$, $|\,|\xi| - |\tau|\,| \sim 2^{-j}$ respectively.

By (19) it thus suffices to show that

$$C^-(A_0, A_0, \Delta_j^+ \Delta_{j+m}^-) \lesssim 2^{(\beta_0+\beta_-+\beta_+-\varepsilon)j} 2^{(\beta_- - \varepsilon)m}$$

uniformly in j, m.

Fix j,m. Partition A_0 into disjoint sectors Γ with radial width 1 and angular width $C^{-1} 2^{-j-\frac{m}{2}}$. We use the notation of Corollary 3.11 with $D = \Delta_j^+ \Delta_{j+m}^-$. From (14) we see that $\Gamma^1 \sim \Gamma^2$ only occurs when the two sectors $\Gamma^1, -\Gamma^2$ subtend an angle of $\sim 2^{-j-\frac{m}{2}}$. By Corollary 3.11 it thus suffices to show that

$$C^-(\Gamma^1, \Gamma^2, \Delta_j^+ \Delta_{j+m}^-) \lesssim 2^{(\beta_0+\beta_-+\beta_+-\varepsilon)j} 2^{(\beta_- - \varepsilon)m}$$

uniformly for $\Gamma^1 \sim \Gamma^2$. By a rotation we may assume that

$$\Gamma^1 = \{\xi : \xi_1 \sim 1, \xi_2 \sim 2^{-j-\frac{m}{2}}\}, \quad \Gamma^2 = \{\eta : \eta_1 \sim -1, \eta_2 \sim 2^{-j-\frac{m}{2}}\}.$$

We now partition Γ^1 and Γ^2 further into disjoint rectangles Q^1, Q^2 which have dimensions $2^{-j} \times 2^{-j-\frac{m}{2}}$. We observe that $Q^1 \sim Q^2$ only when $Q^1 + Q^2 \subset B(0, C2^{-j})$. Thus by Corollary 3.11 again, we reduce to showing that

$$C^-(Q^1, Q^2, \Delta_j^+ \Delta_{j+m}^-) \lesssim 2^{(\beta_0+\beta_-+\beta_+-\varepsilon)j} 2^{(\beta_- - \varepsilon)m}.$$

for $Q^1 \sim Q^2$; by Lemma 3.7 we may assume

$$Q^1 = \{\xi : \xi_1 = 1 + O(2^{-j}), \xi_2 \sim 2^{-j-\frac{m}{2}}\}, \quad Q^2 = \{\eta : \eta_1 = -1 + O(2^{-j}), \eta_2 \sim 2^{-j-\frac{m}{2}}\}.$$

The projection Δ_{j+m}^- is given by a standard multiplier on the region given by (22), so we may discard this projection together with the Δ_j^+ multiplier by (20). Since the α_1, α_2 indices can be set arbitrarily by Lemma 3.13, we write our estimate as

$$C^-_{1/2,1/2}(Q^1, Q^2, 1) \lesssim 2^{(\beta_0+\beta_-+\beta_+-\varepsilon)j} 2^{(\beta_- - \varepsilon)m}.$$

We now eliminate the m parameter by applying the conformal linear transformation

$$L(\xi_1, \xi_2, \tau) = (\frac{\tau + \xi_1}{2} - 2^m \frac{\tau - \xi_1}{2}, 2^{\frac{m}{2}} \xi_2, \frac{\tau + \xi_1}{2} + 2^m \frac{\tau - \xi_1}{2}).$$

By Corollary 3.9 and a computation involving a Taylor expansion of $|\xi|, |\eta|$, we have

$$C^-_{1/2,1/2}(Q^1, Q^2, 1) = |\det L|^{\frac{1}{q} - \frac{1}{3}} C^-_{1/2,1/2}(Q^1_{II_j}, Q^2_{II_j}, 1)$$

where $Q^1_{II_j}, Q^2_{II_j}$ were defined in the model case II_j. We simplify this using $|\det L| = C2^{\frac{3m}{2}}$ and (6) as

$$C^-_{1/2,1/2}(Q^1_{II_j}, Q^2_{II_j}, 1) \lesssim 2^{(\alpha_1+\alpha_2+\frac{3}{q}-2-\varepsilon)j} 2^{(\beta_- - \frac{3}{2q}+\frac{1}{2}-\varepsilon)m}.$$

By (7) it suffices to verify this for $m = 0$:

$$C^-_{1/2,1/2}(Q^1_{II_j}, Q^2_{II_j}, 1) \lesssim 2^{(\alpha_1+\alpha_2+\frac{3}{q}-2-\varepsilon)j}.$$

This is essentially what one would get by specializing (5) to the case II_j. As in the previous section we may assume that $j \gg 1$.

We will treat this using a further conformal transformation

$$\tilde{L}(\xi_1, \xi_2, \tau) = (\frac{\tau+\xi_1}{2} - 2^{2j}\frac{\tau-\xi_1}{2}, 2^j\xi_2, \frac{\tau+\xi_1}{2} + 2^{2j}\frac{\tau-\xi_1}{2}).$$

By Corollary 3.9 we then have

$$C^-_{1/2,1/2}(Q^1_{II_j}, Q^2_{II_j}, 1) \lesssim |\det \tilde{L}|^{\frac{1}{q}-\frac{1}{3}} C^-_{1/2,1/2}(\tilde{Q}^1, \tilde{Q}^2, 1)$$

where $\tilde{Q}^1$, $\tilde{Q}^2$ are the regions

$$\tilde{Q}^1 = \{\xi : \xi_2 \sim 1, |\xi| + \xi_1 = 2 + O(2^{-j})\}$$
$$\tilde{Q}^2 = \{\eta : \eta_2 \sim 1, |\eta| - \eta_1 = 2 + O(2^{-j})\}.$$

Since $|\det \tilde{L}| = C2^{3j}$, we thus reduce to

$$C^-_{1/2,1/2}(\tilde{Q}^1, \tilde{Q}^2, 1) \lesssim 2^{(\alpha_1+\alpha_2-1-\varepsilon)j}. \tag{32}$$

These two sets can be each broken up into about 2^j squares R^1, R^2 of sidelength 2^{-j}. Because these sets lie on angle-separated subsets of the cone, the hypotheses of Lemma 3.12 apply, and we have

$$C^-_{1/2,1/2}(\tilde{Q}^1, \tilde{Q}^2, 1) \lesssim (2^{2j})^{\frac{1}{q}-\frac{1}{2}} \sup_{R^1,R^2} C^-_{1/2,1/2}(R^1, R^2, 1). \tag{33}$$

By Corollary 3.9 we may choose

$$R^1 = R^2 = \{\xi : \xi_1 = 1 + O(2^{-j}), \xi_2 \sim 2^{-j}\}.$$

By repeating the proof of (31) (with the $(+-)$ sign instead of $(++)$) we have

$$C^-_{1/2,1/2}(R^1, R^2, 1) \lesssim 2^{2j(\frac{1}{q}-\frac{1}{2})} 2^{-3j} 2^{5j/q}.$$

Combining this with (33) and simplifying, we have

$$C^-_{1/2,1/2}(\tilde{Q}^1, \tilde{Q}^2, 1) \lesssim 2^{(\frac{9}{q}-5)j},$$

and (32) follows from (12).

As in the previous section, there is an alternate argument which is superior for small q. By replacing $\tilde{Q}^1$, $\tilde{Q}^2$ by their $O(1)$-neighbourhoods and using Proposition 3.6, we obtain

$$C^-_{1/2,1/2}(\tilde{Q}^1, \tilde{Q}^2, 1) \lesssim 1,$$

which gives (32) if $\alpha_1 + \alpha_2 > 1$. This is superior to the previous argument when $q < 9/5$, and for $q = 5/3$ it gives an almost sharp result.

3.17. The contribution of $I - P_\pm$. Let f, g have Fourier transforms supported on A_0, A_j respectively. By considering the $(++)$, $(+-)$ cases separately, we see that one has

$$|\xi| \sim |\xi| + |\tau| \sim 2^j$$

on the region given by (22). Thus by (20) and standard multiplier calculus, the multipliers D^+, D_0 may be replaced by the constant 2^j. By Lemma 3.13 we thus reduce to

$$C^{\pm}_{1/2,1/2}(A_0, A_j, (I - P_\pm)D_-^{\beta_-}) \lesssim 2^{-(\beta_0+\beta_++\frac{1}{2}-\alpha_2+\varepsilon)j},$$

which we write using (6) as

$$C^{\pm}_{1/2,1/2}(A_0, A_j, (I - P_\pm)D_-^{\beta_-}) \lesssim 2^{(\beta_-+\frac{3}{2}-\frac{3}{q}-\alpha_1-\varepsilon)j}.$$

From (14), (15) we see that the value of the multiplier for D_- cannot exceed 1. Thus we may write

$$D_-^{\beta_-} = \sum_{m\geq 0} 2^{-m\beta_-}\Delta^-_m$$

on the region given by (22), where Δ^-_m was defined in the previous section. Discarding the harmless $I - P_\pm$ multiplier by (20) and using (19), it thus suffices to show that

$$C^{\pm}_{1/2,1/2}(A_0, A_j, \Delta^-_m) \lesssim 2^{(\beta_--\varepsilon)m}2^{(\beta_-+\frac{3}{2}-\frac{3}{q}-\alpha_1-\varepsilon)j}$$

uniformly in $m \geq 0$.

Fix j,m. Partition A_0 (resp. A_j) into disjoint sectors Γ^1 (resp. Γ^2) with radial width 1 (resp. 2^j) and angular width $C^{-1}2^{-\frac{m}{2}}$. We use the notation of Corollary 3.11 with $D = \Delta^-_{m-j}$. From (14),(15) we see that $\Gamma^1 \sim \Gamma^2$ only occurs when Γ^1, $\pm\Gamma^2$ differ in the angular variable by $\sim 2^{-m/2}$. Thus by Corollary 3.11 it suffices to show that

$$C^{\pm}_{1/2,1/2}(\Gamma^1, \Gamma^2, \Delta^-_m) \lesssim 2^{(\beta_--\varepsilon)m}2^{(\beta_-+\frac{3}{2}-\frac{3}{q}-\alpha_1-\varepsilon)j}$$

uniformly for $\Gamma^1 \sim \Gamma^2$. The operator Δ^-_m is given by a standard multiplier when restricted to the region given by (22), so we may discard it by (20). By a rotation we may assume that

$$\Gamma^1 = \{\xi : \xi_1 \sim 1, |\xi_2| \ll 2^{-\frac{m}{2}}\}, \quad \Gamma^2 = \{\eta : \eta_1 \sim 2^j, \eta_2 \sim 2^{j-\frac{m}{2}}\}.$$

We now apply Corollary 3.9 with the conformal linear transformation

$$L(\xi_1, \xi_2, \tau) = (\frac{\tau+\xi_1}{2} - 2^m\frac{\tau-\xi_1}{2}, 2^{\frac{m}{2}}\xi_2, \frac{\tau+\xi_1}{2} + 2^m\frac{\tau-\xi_1}{2})$$

to obtain (after applying another mild conformal transformation)

$$C^{\pm}_{1/2,1/2}(\Gamma^1, \Gamma^2, 1) \sim |\det L|^{\frac{1}{q}-\frac{1}{3}} C^{\pm}_{1/2,1/2}(Q^1_{III_j}, Q^2_{III_j}, 1)$$

where $Q^1_{III_j}, Q^2_{III_j}$ were defined in the model case III_j. Since $|\det L| = C2^{3m/2}$, we thus reduce to showing that

$$C^{\pm}_{1/2,1/2}(Q^1_{III_j}, Q^2_{III_j}, 1) \lesssim 2^{(\beta_--\frac{3}{2q}+\frac{1}{2}-\varepsilon)m}2^{(\beta_-+\frac{3}{2}-\frac{3}{q}-\alpha_1-\varepsilon)j}.$$

By (7) it suffices to consider the case $m = 0$. By Lemma 3.13 this becomes

$$C^{\pm}_{0,0}(Q^1_{III_j}, Q^2_{III_j}, 1) \lesssim 2^{(\beta_- + 2 - \frac{3}{q} - \alpha_1 - \varepsilon)j}. \tag{34}$$

This should be compared with condition (8). This estimate is essentially what one would get by specializing (5) to the case III_j. As in the previous sections we may assume $j \gg 1$.

It would be convenient if the two sets $Q^1_{III_j}, Q^2_{III_j}$ were of the same size. Accordingly, we partition $Q^2_{III_j}$ into about 2^{2j} squares Q^2 of sidelength 1, and partition $Q^1_{III_j}$ into the singleton partition $\{Q^1_{III_j}\}$. Because of the angular separation between $Q^1_{III_j}$ and $Q^2_{III_j}$, we see that the conditions of Lemma 3.12 are satisfied, and so we have

$$C^{\pm}_{0,0}(Q^1_{III_j}, Q^2_{III_j}, 1) \lesssim 2^{2j(\frac{1}{q} - \frac{1}{2})} \sup_{Q^2} C^{\pm}_{0,0}(Q^1_{III_j}, Q^2, 1). \tag{35}$$

By Corollary 3.9 and Lemma 3.13 we may move Q^2 to

$$Q^2 = \{\eta : \eta_1 = \mp 2^j + O(1), \eta_2 = O(1)\}$$

without seriously affecting the position of $Q^1_{III_j}$.

We would like to now apply Proposition 3.6. Unfortunately, the surface $S^{\pm}(Q^2)$ is too flat to be of disjoint conic type uniformly in j, and we must make another decomposition. Partition $Q^1_{III_j}$ into about 2^j sets of the form

$$Q^1 = \{\xi \in Q^1_{III_j} : \xi_1 + |\xi| = a + O(2^{-j})\}$$

where $a \sim 1$, and partition Q^2 trivially as $\{Q^2\}$. From a Taylor expansion of $|\xi|$ we see that the condition of Lemma 3.12 is satisfied, so we have

$$C^{\pm}_{0,0}(Q^1_{III_j}, Q^2, 1) \lesssim 2^{j(\frac{1}{q} - \frac{1}{2})} \sup_{Q^1} C^{\pm}_{0,0}(Q^1, Q^2, 1). \tag{36}$$

By Lemma 3.7 we may set $a = 2$ without seriously affecting the position of Q^2.

From (34), (35), (36), we thus reduce to

$$C^{\pm}_{0,0}(Q^1, Q^2, 1) \lesssim 2^{-3j(\frac{1}{q} - \frac{1}{2})} 2^{(\beta_- + 2 - \frac{3}{q} - \alpha_1 - \varepsilon)j}.$$

By (18) it therefore suffices to show

$$\|\widehat{f d\sigma_1} \widehat{g d\sigma_2}\|_q \lesssim 2^{-3j(\frac{1}{q} - \frac{1}{2})} 2^{(\beta_- + 2 - \frac{3}{q} - \alpha_1 - \varepsilon)j} \|f\|_2 \|g\|_2 \tag{37}$$

for f, g supported on $S^+(Q^1)$, $S^{\pm}(Q^2)$ respectively. In null co-ordinates the surfaces become

$$S^+(Q^1) = \{(a_1, a_2, a_3) : a_1 = 1 + O(2^{-j}), a_2 = O(1), a_3 = \frac{a_2^2}{4a_1}\}$$

$$S^{\pm}(Q^2) = \{(a_1, a_2, a_3) : a_3 \sim \pm 2^j + O(1), a_2 = O(1), a_1 = \frac{a_2^2}{4a_3}\}$$

We now apply the non-conformal scaling

$$L(a_1, a_2, a_3) = (2^j a_1, a_2, a_3).$$

The surfaces

$$L(S^+(Q^1)) = \{(a_1, a_2, a_3) : a_1 = 2^j + O(1), a_2 = O(1), a_3 = \frac{a_2^2}{2^{-j+2}a_1}\}$$

$$L(S^\pm(Q^2)) = \{(a_1, a_2, a_3) : a_3 \sim \pm 2^j + O(1), a_2 = O(1), a_1 = \frac{a_2^2}{2^{-j+2}a_3}\}$$

are essentially non-parallel parabolic cylinders and are of disjoint conic type uniformly in j.

Translating the estimate $R^*(2 \times 2 \to q)$ for these surfaces back to $S^+(Q^1)$ and $S^\pm(Q^2)$, we obtain

$$2^{-j/q} \| \widehat{f d\sigma_1} \widehat{g d\sigma_2} \|_q \leq 2^{-j/2} \|f\|_2 \|g\|_2.$$

Combining this with (11) we obtain (37) as desired.

4. Application to Sogge's local smoothing conjecture

We continue the study of one-sided solutions $\phi^\pm = e^{\pm i\sqrt{-\Delta}t} f$ to the free wave equation in $\mathbf{R}^{2+1}$, which we initiated in Section 3.

Figure 5. Local smoothing estimates $LS(p \to q, \alpha)$.

We consider local smoothing estimates of the form

$$\|\phi^\pm\|_{L^q([1,2]\times\mathbf{R}^2)} \lesssim \|f\|_{L^p_\alpha},$$

where $1 \leq p, q \leq \infty$ and $\alpha \in \mathbf{R}$. We call such an estimate $LS(p \to q, \alpha)$. By time reversal symmetry it suffices to consider the solution ϕ^+.

There are several known necessary conditions for $LS(p \to q, \alpha)$ to hold. From translation invariance considerations we have $q \geq p$. From the focussing example in which f is a function of height 1 adapted to a δ-neighbourhood of the circle of radius r for some $r \in [1,2]$, and $\phi^\pm$ has size $\delta^{-1/2}$ on the δ-ball centered at $(r,0)$, we obtain the condition

$$\frac{3}{q} - \frac{1}{2} \geq \frac{1}{p} - \alpha. \tag{38}$$

From the Knapp example, in which $\phi^\pm(x,t)$ for $|t| \lesssim 1$ is essentially a bump of magnitude 1 supported on a $\delta \times \delta^{1/2}$ rectangle, we obtain the condition

$$\frac{3}{2q} \geq \frac{3}{2p} - \alpha. \tag{39}$$

Finally there is the condition

$$\frac{1}{q} + \frac{1}{2} \geq \frac{2}{p} - \alpha$$

which comes from considering the case when f is a delta function.

We display these conditions, and the ranges for which they are dominant, in Figure 5. The endpoints A, H, I correspond to the estimates $LS(1 \to 1, 1/2+\varepsilon)$, $LS(\infty \to \infty, 1/2+\varepsilon)$, $LS(1 \to \infty, 3/2+\varepsilon)$, with the epsilon loss being necessary. The endpoint B is the energy estimate $LS(2 \to 2, 0)$. These four estimates, and the ones obtained by interpolation between them, are the best that one can do from fixed-time estimates alone; any further estimate represents some gain in smoothness by averaging over time, hence the term "local smoothing".

The optimal local smoothing estimate would be $LS(4 \to 4, \varepsilon)$, which is point D in Figure 5. This is the local smoothing conjecture of Sogge, which is still open, although Mockenhaupt, Seeger, and Sogge [21] have the partial result $LS(4 \to 4, 1/8+\varepsilon)$. We improve upon this result in Section 6.

By interpolating the conjecture D with the known estimate I, we obtain

Conjecture 4.1. [28] *We have $LS((q/3)' \to q, 3/2 - 6/q + \varepsilon)$ for all $q \geq 4$ and $\varepsilon > 0$.*

This conjecture is true for $q \geq 6$ thanks to Strichartz' estimate $LS(2 \to 6, 1/2)$, which is the point G. In [28] Schlag and Sogge improved this to $q \geq 5$, which is the point F. We do not know if the ε is necessary for $q > 4$, although the Besicovitch-Rado-Kinney construction and an example in Wolff [37] both show that it is necessary when $q = 4$.

The purpose of this section is to improve this result to

Theorem 4.2. *Suppose that $q_0 < 2$ is such that (5) is true in the $(++)$ case for $q = q_0$, $\beta_0 = \beta_+ = 0$, $\beta_- = \frac{3}{2q_0} - \frac{1}{2} + 2\varepsilon$, and $\alpha_1 = \alpha_2 = \frac{3}{4q_0'} + \varepsilon$ for arbitrarily small $\varepsilon > 0$. Then Conjecture 4.1 is true for all $q \geq q_0 + 3$.*

From Theorem 1.2 and Theorem 3.2 we see that the hypotheses of this theorem hold for $q_0 = 2 - \frac{8}{121} + \varepsilon$. Thus Conjecture 4.1 is true for $q > 5 - \frac{8}{121}$. Unfortunately, since (5) can only hold when $q_0 \geq 5/3$ we see that this theorem cannot prove the full conjecture. We discuss an alternative approach in Section 6.

Proof Fix q_0; without loss of generality we may assume that $q = 3 + q_0$. Write $p = (q/3)'$ and $\alpha = 3/2 - 6/q + \varepsilon$.

Roughly speaking, the idea is as follows. The hypothesis is a bilinear variant of the (false) estimate $LS(2 \to 2q, \tilde{\alpha})$, where $\tilde{\alpha}$ is given by (38). Strictly speaking this estimate is false because of the Knapp condition (39), however the bilinear version (5) can still hold, and this turns out to be good enough for interpolation purposes (as long as the final estimate is on the critical line $p = (q/3)')$. Interpolating this estimate, which corresponds to C on Figure 5, with the point H, we obtain the desired result E. This is a variant of the argument in [28], which used the $q = 2$ estimate, although the version of H used in [28] is slightly different. The approach should also be compared with the sharp restriction theorems proven in [34, 35].

By Littlewood-Paley decomposition and giving an epsilon up in the α index, we may assume that f has frequency support on the annulus $|\xi| \sim N$ for some N; we may assume that $N \gg 1$ since in the low frequency case $\phi^\pm$ is given by a smoothing operator applied to f.

We have to prove

$$\|\phi^+\|_{L^q([1,2]\times\mathbf{R}^2)} \lesssim N^\alpha \|f\|_{L^p}.$$

Since $q > p$ we can use translation invariance and finite speed of propagation to replace $[1, 2] \times \mathbf{R}^2$ by $[1, 2] \times B(0, 1)$. We square this as

$$\|\phi^+\phi^+\|_{L^{q/2}([1,2]\times B(0,1))} \lesssim N^{2\alpha}\|f\|_{L^p}\|f\|_{L^p}.$$

For each $j > 0$, we divide up the circle into 2^j arcs τ^j_k of length 2^{-j}, and partition $f = \sum_k f_{j,k}$, $\phi^+ = \sum_k \phi^+_{j,k}$, where $\widehat{f_{j,k}}$ is supported in the sector $\{\xi : |\xi| \sim N, \xi/|\xi| \in \tau^{j,k}\}$. By the Whitney decomposition (cf. the proof of Proposition 2.3 or [34]) we have

$$\phi^+\phi^+ = \sum_j \sum_{k\sim k'} \phi^+_{j,k}\phi^+_{j,k'}$$

so it suffices to show that

$$\|\sum_{k\sim k'} \phi^+_{j,k}\phi^+_{j,k'}\|_{L^{q/2}([1,2]\times B(0,1))} \lesssim N^{2\alpha}2^{-\varepsilon j}\|f\|_{L^p}\|f\|_{L^p} \tag{40}$$

uniformly in j for some $\varepsilon > 0$.

Fix j. We see that $\phi^+_{j,k}\phi^+_{j,k'}$ are supported in essentially disjoint boxes as $k \sim k'$ both vary. Since $q/2 > 2$, we may apply Lemma 7.1 and estimate the left-hand side of (40) by

$$\left(\sum_{k\sim k'} \|\phi^+_{j,k}\phi^+_{j,k'}\|^r_{L^{q/2}([1,2]\times B(0,1))}\right)^{1/r}$$

where $r = (q/2)'$.

Suppose for the moment that we could prove the estimate

$$\|\phi^+_{j,k}\phi^+_{j,k'}\|_{L^{q/2}([1,2]\times B(0,1))} \lesssim N^{2\alpha}2^{-\varepsilon j}\|f_{j,k}\|_{L^p}\|f_{j,k'}\|_{L^p} \tag{41}$$

for all $k \sim k'$. Then the left-hand side of (40) would be bounded by

$$N^{2\alpha}2^{-\varepsilon j}\left(\sum_{k\sim k'} (\|f_{j,k}\|_{L^p}\|f_{j,k'}\|_{L^p})^r\right)^{1/r}.$$

Since $q > 4$ and $p = (q/3)'$, we have $r \geq p/2$, so the above expression is majorized by

$$N^{2\alpha}2^{-\varepsilon j}\left(\sum_{k\sim k'}(\|f_{j,k}\|_{L^p}\|f_{j,k'}\|_{L^p})^{p/2}\right)^{2/p}.$$

By Cauchy-Schwarz and the fact that there are only finitely many k' associated to each k, we may bound this by

$$N^{2\alpha}2^{-\varepsilon j}\left(\sum_{k}\|f_{j,k}\|_{L^p}^p\right)^{1/p}\left(\sum_{k'}\|g_{j,k'}\|_{L^p}^p\right)^{1/p},$$

which is (40).

Thus it suffices to prove (41). By a rotation it suffices to show that

$$\|\phi^+\psi^+\|_{L^{q/2}([1,2]\times B(0,1))}\lesssim N^{2\alpha}2^{-\varepsilon j}\|f\|_p\|g\|_p \tag{42}$$

whenever f, g have frequency support on the sectors

$$\{\xi : \xi_1\sim N, |\xi_2|\ll 2^{-j}N\},$$
$$\{\eta : \eta_1\sim N, \eta_2\sim 2^{-j}N\}$$

respectively; here the notation is as in Section 3.

We first show this in the easy case when $2^j>\sqrt{N}$. By Hölder's inequality it suffices to show that

$$\|\phi^+\|_{L^q([1,2]\times B(0,1))}\lesssim N^{\alpha}2^{-\varepsilon j}\|f\|_p.$$

We use the integral representation

$$\phi^+(t,x)=\int e^{2\pi i x\cdot\xi}e^{2\pi i t|\xi|}\hat{f}(\xi)\,d\xi.$$

From the support of f and the restrictions $t\sim 1, 2^j>\sqrt{N}$ we have the Taylor approximation

$$t|\xi|=t\xi_1+O(1),$$

where the $O(1)$ error is smooth. Discarding this error[3] it thus suffices to show that

$$\|\int e^{2\pi i x\cdot\xi}e^{2\pi i t\xi_1}\hat{f}(\xi)\,d\xi\|_{L^q([1,2]\times B(0,1))}\lesssim N^{\alpha}2^{-\varepsilon j}\|f\|_p.$$

But by a change of variables and the Fourier inversion formula this becomes

$$\|f\|_{L^q(B(0,C))}\lesssim N^{\alpha}2^{-\varepsilon j}\|f\|_p.$$

But this follows (with $\varepsilon=\alpha/2$) from Lemma 7.3, the frequency support of f, and the definitions of α, p, q.

We now turn to the case when $2^j\lesssim\sqrt{N}$. It suffices to show

$$\|\phi^+\psi^+\|_{L^{q/2}([1,2]\times B(0,1))}\lesssim(2^{-2j}N)^{1-\frac{1}{p}-\frac{3}{q}+\varepsilon}N^{\frac{2}{p}-\frac{3}{q}+\varepsilon}\|f\|_p\|g\|_p, \tag{43}$$

for arbitrarily small ε, since (42) follows from the definitions of p, α, q, the hypothesis $2^j\lesssim\sqrt{N}$, and some algebra. Since $(1/p,1/q)$ is on the line segment between $(1/2,1/2q_0)$ and $(1/\infty,1/\infty)$, it suffices to prove (43) with (p,q) replaced

[3]See [32], Lemma 2.1, [6], Lemma 2.10, or the references therein. A similar argument also appears in the proof of Proposition 5.2 in this paper.

by $(2, 2q_0)$ and (∞, ∞) respectively. Substituting these pairs of exponents into (43) and discarding some epsilons we obtain (after some algebra)

$$\|\phi^+\psi^+\|_{L^{q_0}([1,2]\times B(0,1))} \lesssim N^{\alpha_1} N^{\alpha_2} (2^{-2j}N)^{-\beta_-} \|f\|_2 \|g\|_2$$

and

$$\|\phi^+\psi^+\|_{L^\infty([1,2]\times B(0,1))} \lesssim 2^{-2j} N \|f\|_\infty \|g\|_\infty$$

with the same restrictions on f, g as before, where α_1,α_2, β_- were defined in the statement of the Theorem.

The former estimate is a direct consequence of the hypothesis (5) with the specified values of $q, \beta_0, \beta_+, \beta_-, \alpha_1, \alpha_2$, since D_- is equal to a standard multiplier with height $2^{-2j}N$ on the frequency support of $\phi^+\psi^+$. Thus it suffices to prove the latter estimate. By Hölder's inequality, symmetry it suffices to show that

$$\|\phi^+\|_{L^\infty([1,2]\times B(0,1))} \lesssim N^{1/2} 2^{-j} \|f\|_\infty.$$

It suffices to show this for $2^j = \sqrt{N}$, since in the case $2^j < \sqrt{N}$ can be recovered by decomposing f and using the triangle inequality. But by repeating the Taylor approximation argument used earlier this inequality reduces to $\|f\|_{L^\infty(B(0,C))} \lesssim \|f\|_\infty$, which is of course trivial. ■

5. An improvement to Mockenhaupt's square function estimate

Let N be a large dyadic number, and consider the neighbourhood S_{N^2} of the upper unit light cone

$$S_{N^2} = \{(\xi, \tau) : |\xi| \sim 1, \tau = |\xi| + O(1/N^2)\}.$$

Following [20], we divide this region into N regions

$$E_m = \{(re^{2\pi i\theta}, \tau) : r \sim 1, \theta = \frac{m}{N} + O(\frac{1}{N}), \tau = r + O(1/N^2)\}$$

for $m = 0, \dots, N-1$; note that each E_m is essentially a $1 \times \frac{1}{N} \times \frac{1}{N^2}$ rectangle. Let ψ_m be a bump function essentially adapted to E_m. We will think of the index set $0, \dots, N-1$ as being arranged in a circle, so $N-1$ is adjacent to 0.

In [20], the following square-function estimate was proven for arbitrary test functions f:

$$\|\sum_m \hat{\psi}_m * f\|_4 \lesssim N^{1/4} \|(\sum_m |\hat{\psi}_m * f|^2)^{1/2}\|_4. \tag{44}$$

This estimate has many applications; we will discuss some of these in Section 6.

In [3], it was observed that the $1/4$ exponent in (44) could be improved to $1/4-\tau$ for some $\tau > 0$. The purpose of this section is to refine the argument in [3] and give an explicit value for τ. More precisely, we have

Theorem 5.1. *If $\kappa > 0$ is such that $R^*(2 \times 2 \to 2-\kappa)$ holds for unit-separated subsets of the upper unit light cone and $\tau < \kappa/(16-4\kappa)$, then*

$$\|\sum_m \hat{\psi}_m * f\|_4 \lesssim N^{1/4-\tau} \|(\sum_m |\hat{\psi}_m * f|^2)^{1/2}\|_4, \tag{45}$$

In particular, we have (45) for all $\tau < 1/238$. The best possible value of τ is of course $\tau = 1/4$, but this estimate is well beyond the techniques of this paper. We remark that the best possible value of κ is $1/3$.

Our approach combines the arguments in [3] with the bilinear philosophy of [34], [35] and this paper. As in [34], we begin by reducing the above linear estimate to a bilinear estimate.

Lemma 5.2. *Let $0 < \tau < 1/4$ be fixed. If* (45) *holds, then one has*

(46)
$$\|(\sum_m \hat{\psi}_m * f)(\sum_{m'} \hat{\psi}_{m'} * g)\|_2 \lesssim N^{1/2-2\tau} \|(\sum_m |\hat{\psi}_m * f|^2)^{1/2}\|_4 \|(\sum_{m'} |\hat{\psi}_{m'} * g|^2)^{1/2}\|_4$$

for all f,g where the ranges of m and m' are separated in the sense that dist$(E_m, E_{m'}) \sim 1$ for all m, m'. Conversely, if (46) *holds under this separation condition, then* (45) *holds.*

Proof The former implication is an immediate consequence of Hölder's inequality, so we consider the latter. Assume that (46) holds. To prove (45) it suffices by squaring to estimate the quantity

$$\|(\sum_m \hat{\psi}_m * f)(\sum_m \hat{\psi}_m * f)\|_2;$$

it will be important that we do not conjugate the second factor. For each $1 \leq 2^j \leq N$ we break the interval $[0, N)$ into 2^j dyadic subintervals $\tau_{j,k}$, and write $\tau_{j,k} \sim \tau_{j,k'}$ if $\tau_{j,k}$, $\tau_{j,k'}$ are not adjacent, but have adjacent parents, where we identify 0 with N for the purposes of determining adjacency. When $2^j = N$ we also write $\tau_{j,k} \sim \tau_{j,k'}$ if $\tau_{j,k}$, $\tau_{j,k'}$ are adjacent or equal. We then have the identity (cf. the arguments in Proposition 2.3 and Theorem 4.2)

$$\sum_m \hat{\psi}_m * f \sum_m \hat{\psi}_m * f = \sum_{1 \leq 2^j \leq N} \sum_{\tau_{j,k} \sim \tau_{j,k'}} (\sum_{m \in \tau_{j,k}} \hat{\psi}_m * f)(\sum_{m' \in \tau_{j,k'}} \hat{\psi}_{m'} * f).$$

By the triangle inequality it suffices to consider the contribution of a single j, providing we obtain an exponential gain in j. More precisely, we will show

(47)
$$\begin{aligned} &\|\sum_{\tau_{j,k} \sim \tau_{j,k'}} (\sum_{m \in \tau_{j,k}} \hat{\psi}_m * f)(\sum_{m' \in \tau_{j,k'}} \hat{\psi}_{m'} * f)\|_2 \\ &\qquad \lesssim (2^{-j} N)^{1/2-2\tau} (\|(\sum_m |\hat{\psi}_m * f|^2)^{1/2}\|_4)^2 \end{aligned}$$

We first consider the easy case when $2^j = N$. In this case the only pairs m, m' which appear are those such that $|m - m'| \lesssim 1$, where the norm $|m - m'|$ is taken in the obvious circular sense. By polarization the left-hand side of (47) is therefore majorized by

$$\|\sum_m |\hat{\psi}_m * f|^2\|_2 = (\|(\sum_m |\hat{\psi}_m * f|^2)^{1/2}\|_4)^2$$

as desired.

It remains to consider the case when $2^j < N$. Again, we observe from elementary geometry that the frequency supports of

$$(\sum_{m\in\tau_{j,k}} \hat{\psi}_m * f)(\sum_{m'\in\tau_{j,k'}} \hat{\psi}_{m'} * f)$$

are essentially disjoint as $\tau_{j,k}, \tau_{j,k'}$ vary. Thus we may estimate (47) by

$$(\sum_{\tau_{j,k}\sim\tau_{j,k'}} \|(\sum_{m\in\tau_{j,k}} \hat{\psi}_m * f)(\sum_{m'\in\tau_{j,k'}} \hat{\psi}_{m'} * f)\|_2^2)^{1/2}. \tag{48}$$

We now claim that (46) implies that

$$\begin{aligned} &\|(\sum_{m\in\tau_{j,k}} \hat{\psi}_m * f)(\sum_{m'\in\tau_{j,k'}} \hat{\psi}_{m'} * g)\|_2 \\ &\lesssim (2^{-j}N)^{1/2-2\tau}\|(\sum_{m\in\tau_{j,k}} |\hat{\psi}_m * f|^2)^{1/2}\|_4\|(\sum_{m'\in\tau_{j,k'}} |\hat{\psi}_{m'} * g|^2)^{1/2}\|_4 \end{aligned} \tag{49}$$

for all f,g. When $j = 0$ this is just (46). Now consider the case $j > 0$. By a rotation we may assume that the index sets $\tau_{j,k}$, $\tau_{j,k'}$ are within $C2^{-j}N$ of the origin, so that the sets E_m, $E_{m'}$ are within $C2^{-j}$ of the e_1, e_3 plane. We now apply the conformal linear transformation

$$(\xi_1, \xi_2, \tau) \mapsto (\frac{\tau+\xi_1}{2} - 2^{2j}\frac{\tau-\xi_1}{2}, 2^j\xi_2, \frac{\tau+\xi_1}{2} + 2^{2j}\frac{\tau-\xi_1}{2}).$$

One may verify that the norms on both sides of (49) scale the same way under this transformation. Furthermore, the sets E_m, $E_{m'}$ map to very similar sets but with the role of N replaced by $2^{-j}N$. The claim then follows from (46).

Applying (49) to (48) and comparing this with (47), we see that we only need to verify that

$$(\sum_{\tau_{j,k}\sim\tau_{j,k'}} (\|F_{j,k}\|_4\|F_{j,k'}\|_4)^2)^{1/2} \lesssim (\|(\sum_k |F_{j,k}|^2)^{1/2}\|_4)^2 \tag{50}$$

where

$$F_{j,k} = (\sum_{m\in\tau_{j,k}} |\hat{\psi}_m * f|^2)^{1/2}.$$

By polarizing we may majorize the left-hand side of (50) by

$$(\sum_k \|F_{j,k}\|_4^4)^{1/2}.$$

The claim (50) then follows from interchanging the l^4 and L^4 norms and using the inclusion $l^4 \subset l^2$. ■

Thus it suffices to consider the bilinear estimate. (For an earlier partial equivalence between the two estimates, see [3]).

The next step is to use the restriction hypothesis $R^*(2 \times 2 \to q)$ to obtain something very close to (46). More precisely, we have

Proposition 5.3. *If $q < 2$ is such that $R^*(2 \times 2 \to q)$ holds, then*

(51)
$$(\sum_Q \|(\sum_m \hat{\psi}_m * f)(\sum_{m'} \hat{\psi}_{m'} * g)\|^2_{L^q(Q)})^{1/2} \lesssim N^{1/2} \|(\sum_m |\hat{\psi}_m * f|^2)^{1/2}\|_4 \|(\sum_{m'} |\hat{\psi}_{m'} * g|^2)^{1/2}\|_4$$

where the notation is as in Lemma 5.2, *and Q ranges over a partition of $\boldsymbol{R}^2$ into N-cubes.*

Proof We need to control the left-hand side of (51) by a square-function whose summands have frequency support on eccentric rectangles. To do this, we will first control the left-hand side of (51) by a different square-function whose summands have frequency support on disks. To finish up we will use the pointwise estimate in Lemma 7.2.

For this argument it will be notationally convenient to parameterize frequency space by spatial variables (ξ_1, ξ_2, ξ_3) rather than spacetime variables (ξ_1, ξ_2, τ), and similarly for physical space.

We subdivide frequency space $\mathbf{R}^3$ into a equally spaced collection of $1/N$ cubes I. Each set E_m is covered by about N of these cubes I, with different E_m being covered by essentially disjoint collections of cubes I. We abuse notation and write $I \subset E_m$ if I forms part of the cover of E_m.

We may therefore write for each m

$$\hat{\psi}_m * f(x) = \sum_{I \subset E_m} \hat{\phi}_I * f_m(x)$$

where ϕ_I is a bump function adapted to I and $f_m = \hat{\psi}_m * f$. We can arrange matters so that $\phi_I(\xi) = \phi(\xi - \xi_I)$, where ξ_I is the center of I.

We consider the contribution of a single cube Q in (51). To begin with, let us assume that Q contains the origin.

We have

$$\sum_{I \subset E_m} \hat{\phi}_I * f_m(x) =$$
$$\sum_{I \subset E_m} \int e^{-2\pi i x \cdot \xi} \phi_I(\xi) \check{f}_m(\xi)\, d\xi = e^{-2\pi i x \cdot \xi_I} \int e^{-2\pi i x \cdot (\xi - \xi_I)} \phi(\xi - \xi_I) \check{f}_m(\xi)\, d\xi.$$

As in [35], we perform a Taylor expansion of the phase

$$e^{-2\pi i x \cdot (\xi - \xi_I)} = \sum_\gamma \frac{1}{\gamma!} (\frac{-2\pi i x}{N})^\gamma (N(\xi - \xi_I))^\gamma$$

where γ is a multiindex. The term $\gamma = 0$ should be viewed as the main term. We thus have

$$\sum_m \sum_{I \subset E_m} \hat{\phi}_I * f_m(x) = \sum_\gamma \frac{1}{\gamma!} (\frac{-2\pi i x}{N})^\gamma \sum_m \sum_{I \subset E_m} e^{-2\pi i x \cdot \xi_I} \int \phi^\gamma(\xi - \xi_I) \check{f}_m(\xi)\, d\xi$$

where

$$\phi^\gamma(\xi) = (N\xi)^\gamma \phi(\xi).$$

A similar expansion holds for g. We can therefore write the contribution

$$\|(\sum_m \hat{\psi}_m * f)(\sum_{m'} \hat{\psi}_{m'} * g)\|_{L^q(Q)}$$

of Q to (51) as

$$\| \sum_{\gamma_1,\gamma_2} \frac{1}{\gamma_1!\gamma_2!}(\frac{-2\pi i x}{N})^{\gamma_1+\gamma_2} \prod_{t=1}^2 \sum_m \sum_{I \subset E_m} a_{m,I,t,\gamma_t} e^{-2\pi i x\cdot \xi_I}\|_{L^q(Q)},$$

where

$$a_{m,I,t,\gamma_t} = \int \phi^{\gamma_t}(\xi - \xi_I)\check{f}^t_m(\xi)\, d\xi = \widehat{\phi^{\gamma_t}_I} * f^t_m(0).$$

where we make the convention $f^1 = f$, $f^2 = g$ to simplify the notation.

By the triangle inequality and crudely estimating $|\frac{-2\pi i x}{N}|$by some constant C, we can majorize the above by

$$\sum_{\gamma_1,\gamma_2} \frac{C^{|\gamma_1|+|\gamma_2|}}{\gamma_1!\gamma_2!}\|\prod_{t=1}^2 \sum_m \sum_{I \subset E_m} a_{m,I,t,\gamma_t} e^{-2\pi i x\cdot \xi_I}\|_{L^q(Q)}.$$

We now invoke the hypothesis $R^*(2 \times 2 \to q)$, as discretized in Lemma 5.1 of [35] (with $\alpha = 0$, $n = 3$, $R = N$, and$r = 1$), to estimate this by

$$N^2 \sum_{\gamma_1,\gamma_2} \frac{C^{|\gamma_1|+|\gamma_2|}}{\gamma_1!\gamma_2!}\prod_{t=1}^2 (\sum_m \sum_{I \subset E_m} |a_{m,I,t,\gamma_t}|^2)^{1/2}.$$

Strictly speaking one has to generalize Lemma 5.1 of [35] since the points ξ_I are only within $1/N$ of the cone rather than being on the cone itself, but the argument extends trivially.

Expanding the definition of a_{m,I,t,γ_t} we thus have

$$\|(\sum_m \hat{\psi}_m * f)(\sum_{m'} \hat{\psi}_{m'} * g)\|_{L^q(Q)} \lesssim N^2 \sum_{\gamma_1,\gamma_2} \frac{C^{|\gamma_1|+|\gamma_2|}}{\gamma_1!\gamma_2!}\prod_{t=1}^2 (\sum_m \sum_{I \subset E_m} |\widehat{\phi^{\gamma_t}_I} * f^t_m(0)|^2)^{1/2}$$

This formula applies when Q contains the origin. By translation invariance we may generalize it to

$$\|(\sum_m \hat{\psi}_m * f)(\sum_{m'} \hat{\psi}_{m'} * g)\|_{L^q(Q)} \lesssim N^2 \sum_{\gamma_1,\gamma_2} \frac{C^{|\gamma_1|+|\gamma_2|}}{\gamma_1!\gamma_2!}\prod_{t=1}^2 (\sum_m \sum_{I \subset E_m} |\widehat{\phi^{\gamma_t}_I} * f^t_m(y)|^2)^{1/2}$$

where Q is now arbitrary and $y \in Q$. If we average this in L^2 over all y in Q we obtain

$$\|(\sum_m \hat{\psi}_m * f)(\sum_{m'} \hat{\psi}_{m'} * g)\|_{L^q(Q)} \lesssim N^2 N^{-3/2} \sum_{\gamma_1,\gamma_2} \frac{C^{|\gamma_1|+|\gamma_2|}}{\gamma_1!\gamma_2!}\|\prod_{t=1}^2 (\sum_m \sum_{I \subset E_m} |\widehat{\phi^{\gamma_t}_I} * f^t_m|^2)^{1/2}\|_{L^2(Q)}.$$

Summing this in l^2 over all cubes Q and using the triangle inequality, we thus obtain

$$\text{lhs of (51)} \lesssim N^2 N^{-3/2} \sum_{\gamma_1,\gamma_2} \frac{C^{|\gamma_1|+|\gamma_2|}}{\gamma_1!\gamma_2!}\|\prod_{t=1}^2 (\sum_m \sum_{I \subset E_m} |\widehat{\phi^{\gamma_t}_I} * f^t_m|^2)^{1/2}\|_{L^2(\mathbf{R}^n)}.$$

By Hölder's inequality, we may dominate this by

$$N^2 N^{-3/2} \sum_{\gamma_1,\gamma_2} \frac{C^{|\gamma_1|+|\gamma_2|}}{\gamma_1!\gamma_2!} \prod_{t=1}^{2} \|(\sum_m \sum_{I\subset E_m} |\widehat{\phi_I^{\gamma_t}} * f_m^t|^2)^{1/2}\|_4.$$

By Lemma 7.2 and the fact that ϕ^{γ_t} is a bump function with a norm growing at most exponentially in γ_t, we have the pointwise bound

$$\sum_{I\subset E_m} |\widehat{\phi_I^{\gamma_t}} * f_m^t|^2 \lesssim C^{\gamma_t} M|f_m^t|^2.$$

Inserting this bound into the above, and observing that the γ_1, γ_2 summations are just a convergent series of constants, we see that

$$\text{lhs of (51)} \lesssim N^{1/2} \prod_{t=1}^{2} \|(\sum_m M|f_m^t|^2)^{1/2}\|_4.$$

The claim then follows from the vector-valued maximal function inequality of Fefferman and Stein [11]. ■

This estimate does not have the desired gain of $N^{-\tau}$, but it does improve the L^2 control to L^q control, at least on N-cubes[4]. To finish the proof we will need to interpolate this estimate with another estimate which has the desired gain, but only has L^∞ control on N-cubes. We will use a variant of the estimate in [3], namely

Proposition 5.4. *If* (45) *held for some* $\tau \geq 0$*, then, then one has*

$$\begin{aligned}(\sum_Q \|(\sum_m \hat\psi_m * f)(\sum_{m'} \hat\psi_{m'} * g)\|^2_{L^\infty(Q)})^{1/2} \\ \lesssim N^{\frac{1}{4}-\tau+\varepsilon} \|(\sum_m |\hat\psi_m * f|^2)^{1/2}\|_4 \|(\sum_{m'} |\hat\psi_{m'} * g|^2)^{1/2}\|_4\end{aligned} \tag{52}$$

for all test functions f, g, *where* Q *is as in the previous Proposition.*

Proof By Hölder's inequality and symmetry it suffices to prove the linear estimate

$$(\sum_Q \|\sum_m \hat\psi_m * f\|^4_{L^\infty(Q)})^{1/4} \lesssim \sqrt{N^{\frac{1}{4}-\tau}} \|(\sum_m |\hat\psi_m * f|^2)^{1/2}\|_4. \tag{53}$$

The m represent a partition of the set S_{N^2} into N sectors of angular width $1/N$. We introduce a coarser partition, dividing the $1/N$ neighbourhood S_N of the cone into $N^{1/2}$ sectors $\tilde E_l$, of angular width $N^{-1/2}$; to each m we may associate a parent l such that E_m is essentially contained in $\tilde E_l$. We abuse notation and write $E_m \subset \tilde E_l$ as shorthand for saying that $\tilde E_l$ is the parent of E_m.

The left-hand side of (53) then becomes

$$(\sum_Q \|\sum_l f_l\|^4_{L^\infty(Q)})^{1/4} \tag{54}$$

[4] For previous usage of these types of norms in the study of the wave equation, see [33].

where

$$f_l = \sum_{m:E_m \subset \tilde{E}_l} \hat{\psi}_m * f.$$

From the triangle inequality, Hölder's inequality, and the fact that l ranges over a set with cardinality $N^{1/2}$, we have

$$\| \sum_l f_l \|_{L^\infty(Q)} \lesssim \sum_l \|f_l\|_{L^\infty(Q)} \lesssim N^{3/8} (\sum_l \|f_l\|^4_{L^\infty(Q)})^{1/4}.$$

On the other hand, f_l has frequency support in $\tilde{E}_l$, which is a $1 \times N^{-1/2} \times N^{-1}$ rectangle. Thus by the localized form of Lemma 7.3 in the Appendix, we essentially have

$$\|f_l\|_{L^\infty(Q)} \lesssim N^{-3/8} \|f_l\|_{L^4(Q)}.$$

Strictly speaking we also have to have a rapidly decreasing contribution from cubes other than Q on the right-hand side, but we will ignore this irrelevant complication. Combining these estimates we see that we may majorize (54) by

$$(\sum_Q \| \sum_l f_l \|^4_{L^\infty(Q)})^{1/4} \lesssim (\sum_l \|f_l\|_4^4)^{1/4}.$$

Thus to conclude the proof, we need to show that

$$(\sum_l \|f_l\|_4^4)^{1/4} \lesssim \sqrt{N}^{\frac{1}{4}-\tau} \|(\sum_m |\hat{\psi}_m * f|^2)^{1/2}\|_4.$$

By raising both sides to the fourth power and expanding, this is

$$\sum_l \|f_l\|_4^4 \lesssim \sqrt{N}^{1-4\tau} \sum_m \sum_{m'} \|(\hat{\psi}_m * f)(\hat{\psi}_{m'} * f)\|_2^2.$$

The right-hand side is clearly larger than

$$\sum_l \sqrt{N}^{1-4\tau} \sum_{m:E_m \subset \tilde{E}_l} \sum_{m':E_{m'} \subset \tilde{E}_l} \|(\hat{\psi}_m * f)(\hat{\psi}_{m'} * f)\|_2^2.$$

Thus we need only show that

$$\|f_l\|_4^4 \lesssim \sqrt{N}^{1-4\tau} \sum_{m:E_m \subset \tilde{E}_l} \sum_{m':E_{m'} \subset \tilde{E}_l} \|(\hat{\psi}_m * f)(\hat{\psi}_{m'} * f)\|_2^2$$

uniformly in l. Factorizing the right-hand side and taking fourth roots, this is

$$\|f_l\|_4 \lesssim \sqrt{N}^{\frac{1}{4}-\tau} \|(\sum_{m:E_m \subset \tilde{E}_l} |\hat{\psi}_m * f|^2)^{1/2}\|_4. \tag{55}$$

By changing the definition of $\hat{\psi}_m$ slightly, we may replace $\hat{\psi}_m * f$ in (55) with $\hat{\psi}_m * f_l$. By a rotation we may assume that $\tilde{E}_l$ is within $O(1/\sqrt{N})$ of the e_1, e_3 plane. We now repeat the argument in Lemma 5.2, applying the conformal linear transformation

$$(\xi_1, \xi_2, \tau) \mapsto (\frac{\tau + \xi_1}{2} - N\frac{\tau - \xi_1}{2}, \sqrt{N}\xi_2, \frac{\tau + \xi_1}{2} + N\frac{\tau - \xi_1}{2}).$$

This transformation sends the E_m to $1 \times 1/\sqrt{N} \times 1/N$ rectangles distributed in an essentially regular fashion along a large portion of the C/N neighbourhood of the upper unit light cone. Also the norms on the left and right-hand side of (55) scale in the same way. Thus (55) is just a consequence of the hypothesis (45), with N replaced by $\sqrt{N}$. ∎

Proposition 5.4 represents a gain of $\frac{1}{4} + \tau$ over the $N^{1/2}$ estimate (which would follow from the results of [20]). Despite the quite crude estimates used in the above argument, the sharp hypothesis of $\tau = 1/4$ would give a sharp inequality, as can be seen by taking f to be a bump function.

Combining these two Propositions using Hölder's inequality and the identity

$$\frac{1}{2} = (1 - \frac{\kappa}{2})\frac{1}{q} + \frac{\kappa}{2}\frac{1}{\infty}$$

we thus obtain

Corollary 5.5. *If* (45) *held for some* $\tau \geq 0$*, then* (46) *(and hence* (45)*, by Lemma* 5.2*) holds with* τ *replaced by* $\frac{\kappa}{16} + \frac{\kappa\tau}{4} + \varepsilon$*.*

The map

$$\tau \mapsto \frac{\kappa}{16} + \frac{\kappa\tau}{4}$$

is a contraction with fixed point $\tau = \kappa/(16-4\kappa)$. Thus by applying the above corollary a sufficiently large (but finite) number of times, starting with Mockenhaupt's estimate [20], which gives $\tau = 0$ for instance, we obtain the result.

6. Applications of the square function estimate

In this section we apply the square function estimate (45) just proven to various problems in harmonic analysis. The implications of this estimate are fairly well known, so this section will mainly consist of citing references and stating the exponents.

As observed in [3], the estimate (45) can be inserted directly as a replacement for estimate (1.1) in [20] to obtain an improved estimate on the Bochner-Riesz multiplier for the cone, defined by the multiplier

$$m^\alpha(\xi, \tau) = \phi(\tau)(1 - \frac{|\xi|}{\tau})^\alpha_+,$$

where ϕ is a bump function on $[1, 2]$ and $\alpha > 0$. Indeed, we have

Proposition 6.1. [20, 3] *If* (45) *holds for some* τ*, then the Bochner-Riesz operator of order* α *is bounded on* L^4 *when* $\alpha > 1/8 - \tau/2$*.*

Thus we have boundedness on L^4 when $\alpha > 1/8 - 1/476$. Note that the optimal exponent $\tau = 1/4$ in (45) implies the optimal L^4 (and L^p) result for the Bochner-Riesz operator.

Similarly, this estimate can be used to improve on the local smoothing estimate $LS(4 \to 4, 1/8 + \varepsilon)$ proven in [21]. Indeed, we have

Proposition 6.2. *If* (45) *holds for some* τ*, then we have* $LS(4 \to 4, 1/8 - \tau/2 + \varepsilon)$*.*

We sketch the argument very briefly as follows. In the notation of [21], (45) becomes (ignoring negligible errors)

$$\|Ff\|_4 \lesssim 2^{j(\frac{1}{8}-\frac{\tau}{2})}\|(\sum_\nu |F_\nu f_\nu|^2)^{1/2}\|_4.$$

From the Kakeya estimate in [21] and duality we have

$$\|(\sum_\nu |F_\nu f_\nu|^2)^{1/2}\|_4 \lesssim 2^{\varepsilon j}\|(\sum_\nu |f_\nu|^2)^{1/2}\|_4.$$

Finally, from the square function estimates in [9] we have

$$\|(\sum_\nu |f_\nu|^2)^{1/2}\|_4 \lesssim \|f\|_4.$$

Combining all the estimates gives the result.

In particular, we have smoothing for $\alpha > 1/8 - 1/476$. As before, the optimal exponent $\tau = 1/4$ in (45) would imply the optimal local smoothing conjecture.

Finally, we mention an observation of Oberlin, Smith, and Sogge [23, 24] regarding convolution with the helix in $\mathbf{R}^3$:

$$Tf(x) = \int f(x_1 - \cos t, x_2 - \sin t, x_3 - t)\phi(t)\ dt$$

where ϕ is a bump function. It is easy to see that T maps L^4 to the Sobolev space $L^4_{1/6}$, and in [23] it was conjectured that this could be improved to $L^4_{1/4}$.

Proposition 6.3. [24] *If* (45) *holds for some τ, then T maps L^4 to L^4_α for all $\alpha < 1/6 + \tau/3$.*

Note that the Fourier transform of the helix is concentrated near the light cone, so T is really a cone multiplier in disguise.

We thus have smoothing for the helix for $\alpha < 1/6 + 1/714$. A sharp exponent for (45) would imply an essentially sharp result for the helix.

It is extremely likely that all these results extend to multipliers which are singular on surfaces similar to the cone. Based on the arguments of this paper, one sufficient condition seems to be that portions of the surface whose normals differ by 2^{-j} can be parabolically rescaled by an amount $1 \times 2^j \times 2^{2j}$ to be of disjoint conic type. This may not be the most general condition.

7. Appendix: some elementary harmonic analysis

In this section we state some elementary results which were used repeatedly in the paper.

We begin with a well-known quasi-orthogonality property of functions with disjoint frequency support. Define a *rectangle* to be the product of n (possibly half-infinite or infinite) intervals in $\mathbf{R}^n$.

Lemma 7.1. *Let R_k be a collection of rectangles in frequency space such that the dilates $2R_k$ are almost disjoint, and suppose that f_k are a collection of functions whose Fourier transforms are supported on R_k. Then for all $1 \leq p \leq \infty$ we have*

$$(\sum_k \|f_k\|_p^{p^*})^{1/p^*} \lesssim \|\sum_k f_k\|_p \lesssim (\sum_k \|f_k\|_p^{p_*})^{1/p_*},$$

where $p_* = \min(p, p')$, $p^* = \max(p, p')$.

Proof Let P_k be a smooth Fourier multiplier adapted to $2R_k$ which equals 1 on R_k. We claim that

$$\|\sum_k P_k F_k\|_p \lesssim (\sum_k \|F_k\|_p^{p_*})^{1/p_*}$$

$$(\sum_k \|\sum_k P_k F\|_p^{p^*})^{1/p^*} \lesssim \|F\|_p$$

for arbitrary functions F_k, F; the lemma then follows by setting $F_k = P_k F_k = f_k$, $F = \sum_k F_k$.

The latter estimate is dual to the former (with p replaced by p'), so it suffices to prove the former estimate. By interpolation it suffices to prove this estimate for $p = 1$, $p = 2$, and $p = \infty$. When $p = 2$ the estimate is immediate from Plancherel's theorem. When $p = 1$ or $p = \infty$ the lemma follows from the triangle inequality and the estimates

$$\|P_k F_k\|_1 \lesssim \|F_k\|_1, \quad \|P_k F_k\|_\infty \lesssim \|F_k\|_\infty,$$

which follow from Young's inequality and standard estimates on the kernel of P_k. ■

We also recall a pointwise estimate of Rubio de Francia, which is closely related to the standard square function estimate for equally-spaced cubes in frequency space [9], [7].

Lemma 7.2. [25] *Let* $\{I\}$ *be a collection of equally spaced cubes* I, *and let* $\phi_I(\xi) = \phi(\xi - \xi_I)$ *be bump functions adapted to* I, *where* ξ_I *denotes the center of* I. *Then for any function* f *we have the pointwise estimate*

$$(\sum_I |\widehat{\phi_I} * f|^2)^{1/2} \leq C(\phi)(M|f|^2)^{1/2}$$

where M *is the Hardy-Littlewood maximal function, and* $C(\phi)$ *depends only on the dimension* n *and finitely many of the derivatives of* ϕ *(after rescaling* I *to be of unit length).*

We also give a standard variant of Sobolev's inequality.

Lemma 7.3. *Let* $p < q$. *If* f *is a function whose Fourier transform is supported on a rectangle* Q, *then* $\|f\|_q \lesssim |Q|^{1/p-1/q}\|f\|_p$.

Proof We observe that the claim is invariant under affine transformations, so we may take Q to be the unit cube. In this case we may write $f = f * \psi$ where ψ is a Schwarz function whose Fourier transform equals one on the unit cube. The claim then follows from Young's inequality. ■

By using the rapid decay of ψ in the above proof, we see that we may essentially localize this estimate to rectangles q dual to Q. More precisely, we have

$$\|f\|_{L^q(q)} \lesssim \sum_{q'} c_{q,q'} |Q|^{1/p-1/q} \|f\|_{L^p(q')}$$

where q' ranges over a disjoint set of rectangles dual to Q, and $c_{q,q'}$ is rapidly decreasing for q' away from q.

Note that Hölder's inequality provides a "dual" to the above lemma: If f is a function supported in a set E, then $\|f\|_p \lesssim |E|^{1/p-1/q}\|f\|_q$.

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