ABSTRACT. In this paper, we continue the study of three-dimensional bilinear restriction and Kakeya estimates which was initiated in [17]. In particular, we give new linear and bilinear restriction estimates for the cone, sphere, and paraboloid in $\mathbf{R}^3$, building upon and unifying previous work in this direction by Bourgain, Wolff, and others. In a subsequent paper [18] we will give applications of these estimates to some open problems in harmonic analysis and wave equations.

# A BILINEAR APPROACH TO CONE MULTIPLIERS I. RESTRICTION ESTIMATES

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## 1. Introduction and Notation

Let $S_1$ and $S_2$ be two smooth compact hypersurfaces with boundary in $\mathbf{R}^3$, with Lebesgue measure $d\sigma_1$ and $d\sigma_2$ respectively. If $0 < p, q \le \infty$, we say that the bilinear adjoint restriction estimate $R^*_{S_1,S_2}(p \times p \to q)$ holds if

$$\|\prod_{t=1}^{2} \widehat{f_t d\sigma_t}\|_{L^q(\mathbf{R}^3)} \lesssim \prod_{t=1}^{2} \|f_t\|_p,$$

for all test functions $f_1$, $f_2$ supported on $S_1$, $S_2$ respectively. We also define a local version $R^*_{S_1,S_2}(p \times p \to q, \alpha)$ for $\alpha \ge 0$, by the statement

$$\|\prod_{t=1}^{2} \widehat{f_t d\sigma_t}\|_{L^q(B(0,R))} \lesssim R^{\alpha} \prod_{t=1}^{2} \|f_t\|_p,$$

for all $R \gg 1$.

If $S_1 = S_2 = S$, then the bilinear estimates $R^*_{S_1,S_2}(p \times p \to q)$ and $R^*_{S_1,S_2}(p_1 \times p_2 \to q, \alpha)$ are equivalent to the linear adjoint restriction estimates

$$\|\widehat{f d\sigma}\|_{L^{q/2}(\mathbf{R}^3)} \lesssim \|f\|_p$$

and

$$\|\widehat{f d\sigma}\|_{L^{q/2}(B(0,R))} \lesssim R^{\alpha/2} \|f\|_p$$

respectively; to be consistent with the notation of [17] we denote these estimates by $R^*_S(p \to q/2)$ and $R^*_S(p \to q/2, \alpha/2)$ respectively. These estimates are well understood in two dimensions, but there are many open problems remaining in three and higher dimensions. In this paper we concentrate on the three-dimensional case, although we comment briefly in the four-dimensional case at the end of the paper.

In [17] the bilinear estimates were studied under the assumption that $S_1$, $S_2$ were unit-separated subsets of a graph of an elliptic phase function (see [17], Section 2); these estimates were then used to obtain new progress on the corresponding linear

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estimates. The primary purpose of this paper is to extend these results to more general surfaces, in particular subsets of the light cone in $\mathbf{R}^{2+1}$, mainly by pursuing the ideas in [6].

**Definition 1.1.** Suppose that $S_1$ and $S_2$ are compact surfaces with boundary in $\mathbf{R}^3$. If $\xi \in S_t$ , $t = 1, 2$, we use $n_t(\xi) \in S^2/\pm$ to denote the unit normal to $S_t$ at $\xi$. We say that the pair $S_1$ and $S_2$ are of *disjoint conic type* if the following statements hold:

- (Transversality) For all $\xi_t \in S_t$, $t = 1, 2$ we have $n_t(\xi_t) \in N_t$, where $N_1$ and $N_2$ are small disjoint caps in $S^2/\pm$ which are separated by a distance comparable to 1.
- (Null direction) The map $dn_t : T_{\xi_t} S_t \to T_{\xi_t} S_t$ has eigenvalue 0 with multiplicity one in the direction $w_t(\xi_t) \in S^2/\pm$. We also assume that the remaining eigenvalue has magnitude $\sim 1$.
- (Transversality of null directions) For all $\xi_t \in S_t$, $t = 1, 2$ we have $w_t(\xi_t) \in W_t$, where $W_1$ and $W_2$ are small disjoint caps in $S^2/\pm$ which are separated by a distance comparable to 1. Furthermore, the maximal angular separation between $W_1$ and $N_2$, or $W_2$ and $N_1$, is strictly less than $\pi/2$.

FIGURE 1. Two surfaces of disjoint conic type.

The last condition means that the null directions of $S_1$ (resp. $S_2$) are always transverse to the tangent planes of $S_2$ (resp. $S_1$). In particular, the null directions of $S_1$ and the null directions of $S_2$ are disjoint. Moreover, the first condition, the transversality of the surfaces, is a consequence of this one.

One model example of a pair of surfaces of disjoint conic type are

$$S_t = \{(\underline{\xi}, \xi_3) : \underline{\xi} \in \mathbf{R}^2, \xi_3 = |\underline{\xi}| \sim 1, |\frac{\underline{\xi}}{|\underline{\xi}|} - e_t| \ll 1\},$$

where $e_1$, $e_2$, $e_3$ is the canonical basis for $\mathbf{R}^3$. Another example is when $S_1$ and $S_2$ are subsets of non-parallel cylinders $C_1$ and $C_2$, such that the normals to $S_1$ are separated from the normals to $S_2$, and the null directions of one cylinder are not parallel to the tangent planes of the other.

Our first main result is the following bilinear adjoint restriction theorem for surfaces of disjoint conic type in three dimensions.

**Theorem 1.2.** *If $S_1$ and $S_2$ are of disjoint conic type, then $R^*_{S_1,S_2}(2 \times 2 \to p)$ for all $p > 2 - \frac{8}{121}$.*

This estimate is straightforward for $p = 2$. In [6] this estimate was proven for some $p < 2$; the methods in that paper were not designed with the intent to compute $p$ explicitly, but we conservatively estimate it as $p > 2 - \frac{13}{2408}$.

We prove this theorem in Section 7, after several preliminaries starting in Section 2. The result is certainly not sharp: Machedon and Klainerman have recently conjectured this estimate should hold for $p \geq \frac{5}{3}$. In Section 7 we show why the exponent $\frac{5}{3}$ is best possible.

FIGURE 2. Status of $R^*(p \times p \to q)$ and $R^*(p \to 2q)$ for the cone in $\mathbf{R}^3$.

Note that these estimates are only available in the bilinear setting. The linear estimate $R^*(p \to q)$ is known to hold if and only if $q > 4$ and $q \geq 3p'$ [1], which is a much smaller range of exponents.

We display the above results in Figure 2. The smaller trapezium displays the range of linear estimates $R^*(p \to 2q)$, while the larger trapezium displays the (conjectured) range of bilinear estimates $R^*(p \times p \to q)$. The points $1-7$ represent respectively Strichartz' estimate; Barcelo's estimate [1]; the estimate $R^*(2\times 2 \to 2)$; Theorem 2.1; Bourgain's estimate [6], Theorem 1.2; and the conjecture $R^*(2 \times 2 \to 5/3)$ of Machedon and Klainerman. The dashed line thus represents the best results known to date.

The techniques in this paper can also be applied to surfaces with non-vanishing curvature. Indeed, we have the following new estimates for the three-dimensional linear and bilinear restriction problems in this context.

**Theorem 1.3.** *Let $S$ be a graph of an elliptic phase function (in the sense of* [17] *or* [12, 13]*), and let $S_1$, $S_2$ be unit-separated subsets of $S$. For all $\varepsilon > 0$ and $q > 4 - 8/31$, we have*

$$R^*_S(26/11 + \varepsilon \to 4 - 2/7 + \varepsilon) \tag{1}$$

$$R^*_{S_1,S_2}(2 \times 2 \to 2 - 2/17 + \varepsilon) \tag{2}$$

$$R^*_S((\frac{q}{2})' \to q) \tag{3}$$

The estimate (1) is a new restriction theorem for convex surfaces, improving upon [17], Theorem 4.1 (which in turn improved upon earlier work of [20], [7], [3]), which gave

$$R^*(170/77 + \varepsilon \to 4 - 2/9 + \varepsilon).$$

The estimate (2) represents new progress for the Klainerman-Machedon conjecture for the sphere, and is superior to [17], Corollary 4.6.

The estimate (3) is invariant under parabolic scaling, and thus can be extended to give a restriction theorem for the entire paraboloid. This improves upon [17], Theorem 4.1, which proved the same result for $q > 4-5/27$. (The case $q \geq 4$ follows from the classical Tomas–Stein theorem). We will prove this theorem in Section 8. We also discuss extensions of these results to the negative curvature case.

We remark that the constants in the above estimates only depend on the constant in the definition of the elliptic phase function or disjoint conic type and the $\mathcal{C}^N$ norm of $S_1$ and $S_2$ for some large $N$.

In the sequel to this paper [18] we will show how one can apply the above estimates to the wave and Schrödinger equations in $\mathbf{R}^{2+1}$.

## 2. Overview of proof of Theorem 1.2

Our approach follows broadly the strategy in [6]. At its core, the approach requires two key estimates.

The first key estimate is a good bilinear restriction estimate which captures some residual curvature of $S_1$ and $S_2$ by means of Radon transform $L^p$ estimates:

**Theorem 2.1.** *Suppose that $S_1$ and $S_2$ are of disjoint conic type. Then*

$$R^*_{S_1,S_2}(p \times p \to 2)$$

*holds for all $p \geq \frac{12}{7}$.*

This generalizes (for three dimensions) the corresponding estimate in [17, 13]. The estimate is easy to show for $p = 2$, and works for general $S_1$ and $S_2$ as long as they are transverse. The point is that one can exploit the curvature in the disjoint conic type condition to obtain an estimate for certain $p < 2$; this was first observed by Bourgain [6]. Fortunately these estimates can be proven by the very well-developed theory of $L^p$ estimates for Fourier integral operators.

In [17] the condition on $p$ is shown to be best possible. One should probably be able to prove the above theorem in general dimension $\mathbf{R}^n$, with $\frac{12}{7}$ replaced by $\frac{4n}{3n-2}$, under an appropriate hypothesis which generalizes the three-dimensional disjoint conic type condition.

The second key estimate is a good Kakeya estimate (or more precisely, an x-ray transform estimate) which is associated to $S_1$ and $S_2$.

If $S_t \subset \mathbf{R}^n$ is a hypersurface and $\xi \in S_t$, we let $n_t(\xi)$ denote the normal of $S_t$ at $\xi$, and $L_\xi S_t$ denote the space of all lines $l$ in $\mathbf{R}^n$ which are parallel to $n_t(\xi)$ and intersect the unit ball in $\mathbf{R}^n$; we endow this space with the obvious induced Euclidean metric. Fix $\delta > 0$, and let $\mathcal{E}_i$ be a maximal $\delta$-separated subset of $S_t$, and for each $\xi \in \mathcal{E}_t$, let $L_\xi \mathcal{E}_t$ be a maximal $\delta$-separated subset of lines in $L_\xi S_t$. Let $L\mathcal{E}_t$ denote the bundle of all the $L_\xi \mathcal{E}_t$ over $\mathcal{E}_t$. We endow each $L_\xi \mathcal{E}_t$ with counting measure $di$, and $\mathcal{E}_t$ with normalized counting measure $d\xi$, which is $\delta^{n-1}$ times counting measure.

We define the adjoint discretized x-ray transform $X_t^* = X_{\delta,S_t}^*$ as a map from functions in $L\mathcal{E}_t$ to functions in $\mathbf{R}^n$, as

$$X_t^* F(x) = \sum_{\xi \in \mathcal{E}_t} \sum_{i \in L_\xi \mathcal{E}_t} F(\xi, i) \chi_{T_{\xi,i}^\delta}(x), \tag{4}$$

where $T_{\xi,i}^\delta$ denotes the tube of roughly unit length and thickness $\delta$ centered around the line $i$ and contained in the unit ball.

If $S_1$, $S_2$ are pairs of hypersurfaces and $0 < p, r, q \le \infty$, we use $K_{S_1,S_2}^*((p,r) \times (p,r) \to q)$ to denote the estimate

$$\| \prod_{t=1}^2 X_t^* f_t \|_q \lesssim \delta^{\frac{n}{q} - 2n + 2 - \varepsilon} \prod_{t=1}^2 \| f_t \|_{L_\xi^p L_i^r} \tag{5}$$

for all $f_t$ on $L\mathcal{E}_t$, $t = 1, 2$ and all $\varepsilon > 0$, with the constant depending on $\varepsilon$. To see why the exponent $\delta^{\frac{n}{q} - 2n + 2}$ is sharp, we take $f_t(\xi, i) = \chi_{T_{\xi,i}^\delta}(0)$. Then $X_t^* f_t$ is comparable to $\delta^{-n+1}$ on a ball of volume $\delta^n$.

When $r = 1$ and $S_1$, $S_2$ have non-zero Gaussian curvature, then these estimates are bilinear Kakeya estimates of the form studied in [17]. For instance, we have

$$K_{S_1,S_2}^*\left(\left(\frac{n+2}{n+1}, 1\right) \times \left(\frac{n+2}{n+1}, 1\right) \to \frac{n+2}{2n}\right)$$

in this setting ([17], Theorem 3.4).

However, when $S_1$ and $S_2$ have one vanishing principal curvature, then the estimates are far less favourable. Nevertheless, it was observed by Bourgain [6] that it is still possible to improve upon the standard estimate[1] $K_{S_1,S_2}^*((1,1) \times (1,1) \to 1)$ in this setting, at least in three dimensions. In fact, we have:

[1] See [17], Proposition 3.2.

**Proposition 2.2.** *If $S_1$, $S_2 \subset \boldsymbol{R}^3$ are of disjoint conic type, then $K^*_{S_1,S_2}((r,r) \times (r,r) \to 1)$ for all $1 \leq r \leq 4/3$.*

We will prove this estimate in Section 4, adapting an argument ofBourgain [6], who implicitly proved the above for $1 \leq r \leq 8/7$. We will also show that the exponents in this proposition are sharp. It seems of interest to determine the correct analogue of this estimate in higher dimensions.

We now indicate how these two key estimates are used to prove Theorem 1.2. By interpolating Theorem 2.1 with more elementary estimates, such as $R^*(2 \times 2 \to 1, 1)$ (which is a bilinear form of the Sobolev trace lemma $R^*(2 \to 2, 1/2)$) we can obtain many estimates of the form $R^*(p \times p \to q, \alpha)$ for some relatively large $\alpha$.

Our aim is to eventually obtain a non-trivial estimate with $p = 2$ and $\alpha = 0$. The first step is to use Proposition 2.2 to lower the value of $\alpha$. (The ability to use Kakeya estimates to improve upon local restriction estimates was observed by Bourgain [3, 7, 6]). This will be achieved by inserting Proposition 2.2 into the following lemma:

**Lemma 2.3.** *Let $n \geq 2$, and suppose that $S_1$ and $S_2 \subset \boldsymbol{R}^n$ be compact hypersurfaces. If $2 \leq 2r < p, q < \infty$, $\alpha \geq 0$ are such that $K^*_{S_1,S_2}((\frac{pr}{2}, r) \times (\frac{pr}{2}, r) \to \frac{qr}{2})$ and $R^*_{S_1,S_2}(\frac{2}{r} \times \frac{2}{r} \to q, \alpha)$ hold, then $R^*_{S_1,S_2}(p \times p \to q, \frac{\alpha}{2} + \varepsilon)$ holds for all $\varepsilon > 0$.*

This is a generalization of Lemma 4.4 in [17], which dealt with the case $r = 1$, and also generalizes some arguments implicit in [3, 7, 6]. Note that this lemma does not require any curvature assumptions on $S_1$, $S_2$. We assure the reader that the exponents in the above lemma are natural, as should hopefully become clear in Section 5, in which Lemma2.3 is proved.

Note that the restriction estimate in the conclusion of Lemma 2.3 issuperior to that in the hypothesis in the sense that the $\alpha$ index is improved, but is inferior in the sense that the $\frac{2}{r}$ exponent needed to be upgraded to $p$. It would be very convenient if this inferiority could be somehow removed, as one could then iterate the above argument indefinitely and make $\alpha$ arbitrarily small. Although we cannot do this directly, we may still combine this Lemma and Proposition 2.2 with an interpolation argument involving the estimate in Theorem 2.1. This allows us to partially iterate this argument and get a restriction estimate with a relatively small value of $\alpha$. To remove the $\alpha$ entirely requires an additional argument. Two examples of such arguments are in [7] and [16] respectively; however, we will use an improved version of the argument in Bourgain [6], which we phrase as follows.

**Lemma 2.4.** [6] *Let $n \geq 2$, and let $S_1$ and $S_2$ be compact hypersurfaces with boundary such that the Fourier transforms of $d\sigma_t$, $t = 1, 2$ satisfy the decay estimate*

$$|\widehat{d\sigma_t}(x)| \lesssim (1 + |x|)^{-\sigma} \tag{6}$$

*for some $\sigma > 0$. Then for any $1 < q < \frac{\sigma+1}{\sigma}$, $\alpha > 0$, and $\varepsilon > 0$, the estimate $R^*_{S_1,S_2}(2 \times 2 \to q, \alpha)$ implies the estimate $R^*_{S_1,S_2}(2 \times 2 \to p)$ for all $\frac{1}{p}\big(1 + \frac{2\alpha}{\sigma}\big) < \frac{1}{q} + \frac{\alpha}{1+\sigma}$.*

We prove this lemma in Section 6. In Section 7 we show how all these steps combine together to give Theorem 1.2.

## 3. Proof of Theorem 2.1

By localizing $S_1$ and $S_2$ and applying a rotation if necessary, we may assume that $S_1$ and $S_2$ are graphs whose domains are small balls. By translating $S_1$ and $S_2$ we may assume that these balls are centered at the origin, thus:

$$S_t = \{(x, \Phi_t(x)) : x \in \mathbf{R}^2, |x| \ll 1\}.$$

where $\Phi_t : \mathbf{R}^2 \to \mathbf{R}$ are smooth functions such that $\Phi_t(0) = 0$ for $t = 1, 2$.

We now translate the disjoint conic type conditions in Definition 1.1 into this setting.

The transversality condition implies that

$$|\nabla\Phi_1(x^1) - \nabla\Phi_2(x^2)| \sim 1$$

for all $|x^1|, |x^2| \ll 1$.

The null direction condition implies that the Hessian matrices

$$H\Phi_t(x^t) = \big(\partial_i\partial_j\Phi_t(x^t)\big)_{i,j=1,2}$$

have a zero eigenvalue with multiplicity 1 and another eigenvalue with magnitude $\sim 1$; let $v_t(x^t) \in S^1/\pm$ denote the zero eigenvector. Note that the vector $(v_t(x^t), \partial_{v_t(x^t)}\Phi_t(x^t))$ is parallel to the null principal direction of $S_t$ at $(x^t, \Phi(x^t))$.

Finally, we claim that the transverse null direction condition implies that

$$\langle\nabla\Phi_1(x^1) - \nabla\Phi_2(x^2), v_1(x^1)\rangle \neq 0 \tag{7}$$

for all $|x^1|, |x^2| \ll 1$. For, if (7) did not hold, then

$$(-\nabla\Phi_2(x^2), 1) \cdot (v_1(x^1), \partial_{v_1(x^1)}\Phi_1(x^1)) = 0,$$

so that the null principal direction of $S_1$ at $(x^1, \Phi(x^1))$ is orthogonal to the normal of $S_2$ at $(x^2, \Phi(x^2))$, contradicting the transverse null direction hypothesis. A similar conclusion holds with the roles of $S_1$ and $S_2$ reversed.

After these preliminaries we can now prove Theorem 2.1. By Hölder's inequality, it suffices to show that

$$\int |\widehat{f_1 d\sigma_1}\widehat{f_2 d\sigma_2}|^2 \lesssim \|f_1\|_{12/7}^2\|f_2\|_{12/7}^2.$$

where $f_t$ are supported on $S_t$. Since we are assuming the surfaces $S_t$ to be graphs, we may replace the measures $d\sigma_t$ by

$$\tilde{d}\sigma_t(x_1, x_2, x_3) = \delta(x_3 - \Phi_t(x_1, x_2))\; dx_1 dx_2$$

where $x_1$, $x_2$, $x_3$ are the standard co-ordinates of $\mathbf{R}^3$, since one can absorb any Jacobian factors into the functions $f_t$.

By Plancherel, the integral we want to estimate is equal to

$$\int f_1(x)f_2(y)\overline{f_1(z)f_2(w)}\delta(\Phi_1(x)+\Phi_2(y)-\Phi_1(z)-\Phi_2(w))\delta(x+y-z-w)\; dxdydzdw.$$

We follow the argument in [17]. By multilinear interpolation it suffices to show that

$$\int f_1(x)f_2(y)g_1(z)g_2(w)\delta(\Phi_1(x)+\Phi_2(y)-\Phi_1(z)-\Phi_2(w))$$
$$\delta(x+y-z-w)\ dxdydzdw \lesssim \|f_1\|_1\|f_2\|_{3/2}\|g_1\|_\infty\|g_2\|_{3/2}$$

together with a similar estimate with the subscripts 1,2 reversed. By symmetry it suffices to prove the displayed estimate.

By the reductions in [17] it suffices to show that

$$\| \int f_2(y)\delta(\phi_x(y,w))\psi(y,w)\ dy\,\|_{L^3_w} \lesssim \|f_2\|_{3/2}$$

uniformly in $x$, where $\phi = \phi_x(y,w)$ is the function

$$\phi_x(y,w) = \Phi_1(x)+\Phi_2(y)-\Phi_1(x+y-w)-\Phi_2(w)$$

and $\psi(y,w)$ is a cutoff function. By Lemma 2.9 in [17] (see also [15], page 428) it suffices to show that $\phi$ satisfies the rotational curvature condition

$$rot\,curv\phi = \left|\det\begin{pmatrix} \phi & \phi_y \\ \phi_w & \phi_{yw}\end{pmatrix}\right| > 0 \quad \text{when } \phi = 0 \tag{8}$$

uniformly in all variables.

If the Hessian matrix $H\Phi_1(x)$ has eigenvalues $\mu_1, 0$ we can compute

$$rot\,curv\phi_x(y,y) = \mu_1(\partial_{v_1(x)}\Phi_1(x) - \partial_{v_1(x)}\Phi_2(y))^2.$$

(For instance, consider a coordinate system in which the Hessian matrix is diagonal). From (7) we thus see that $rot\,curv\phi_x(y,w) \neq 0$ if $w$ and $y$ are sufficiently close. The claim then follows by localizing $S_2$, using the compactness of the surface. ∎

## 4. Proof of Proposition 2.2

We begin by observing a geometric consequence of the disjoint conic type hypothesis.

**Lemma 4.1.** *For $\xi_t,\ \xi_t' \in S_t$, let $[n_t(\xi_t), n_t(\xi_t')]$ denote the subspace spanned by $n_t(\xi_t)$ and $n_t(\xi_t')$. If $S_1$ and $S_2$ are of disjoint conic type and their diameters are small enough, then there is a constant $c > 0$ such that for all $\xi_1,\ \xi_1' \in S_1$ and $\xi_2 \in S_2$, we have*

$$\angle([n_2(\xi_2), n_2(\xi_2')], n_1(\xi_1)) \geq c. \tag{9}$$

*A similar statement holds with the roles of $S_1$ and $S_2$ reversed.*

**Proof** By symmetry it suffices to prove (9).

By the transversality of the null directions, there is a small constant $\epsilon > 0$ such that, $\angle(w_2(\xi_2), n_1(\xi_1)) \leq \frac{\pi}{2} - \epsilon$.

By the triangle inequality, it thus suffices to show that

$$\angle(w_2(\xi_2), [n_2(\xi_2), n_2(\xi_2')]) \geq \frac{\pi}{2} - \epsilon/2 \tag{10}$$

if $\xi_2'$ is sufficiently close to $\xi_2$, since we can then take $c = \epsilon/2$.

To see (10), we first observe that the Gauss map $n_2 : S_2 \to S^2/\pm$ is smooth and has constant rank 1 by the null direction hypothesis. This implies (see e.g. [14], Chapter 2, Theorem 9) that the image of $n_2$ is a smooth immersed one-dimensional submanifold $M_2$ of $S^2/\pm$. Since the differential map $dn_2$ is self-adjoint, its image is orthogonal to its null eigenvector $w_2$; since $n_2$ has magnitude 1, the image is also orthogonal to $n_2$. Thus we see that the tangent space of $M_2$ at $n_2(\xi_2)$ is orthogonal to both $n_2(\xi_2)$ and $w_2(\xi_2)$. This implies that

$$n_2(\xi_2') = n_2(\xi_2) + v + O(|v|^2)$$

for all $\xi_2'$ which are sufficiently close to $\xi_2$, where $v$ is a vector orthogonal to $n_2(\xi_2)$ and $w_2(\xi_2)$ with magnitude $\ll \epsilon$. The claim (10) then follows from elementary geometry and the fact that $n_2(\xi_2)$ is orthogonal to $w_2(\xi_2)$. ∎

We remark that this argument extends easily to higher dimensions. We are indebted to Peter Petersen for pointing out some simplifications in the above proof.

Henceforth we will assume (shrinking $S_1$, $S_2$ if necessary) that $S_1$, $S_2$ are such that (9) and its symmetric counterpart both hold.

By interpolation with the estimate $K^*_{S_1,S_2}((1,1) \times (1,1) \to 1)$ it will be enough to prove the end-point estimate $K^*_{S_1,S_2}((4/3,4/3) \times (4/3,4/3) \to 1)$. By symmetry and interpolation (see [2]), this will follow from

$$\int X_1^* f_1(x) X_2^* f_2(x)\, dx \leq \delta^{-1-\epsilon} \|f_1\|_{L^1_\xi(L^2_i)} \|f_2\|_{L^2_\xi(L^1_i)}. \tag{11}$$

By definition,

$$\int X_1^* f_1(x) X_2^* f_2(x)\, dx = \sum_{\xi_1} \sum_{\xi_2} \sum_{i_1} \sum_{i_2} f_1(\xi_1, i_1) f_2(\xi_2, i_2) |T_{\xi_1,i_1} \cap T_{\xi_2,i_2}|.$$

It suffices to prove that

$$\sum_{\xi_2} \sum_{i_1} \sum_{i_2} f_1(\xi_1, i_1) f_2(\xi_2, i_2) |T_{\xi_1,i_1} \cap T_{\xi_2,i_2}| \lesssim \delta^{1-\varepsilon} \|f_1(\xi_1, \cdot)\|_{L^2_i} \|f_2\|_{L^2_\xi(L^1_i)},$$

uniformly in $\xi_1$, since (11) follows by summing in $\xi_1$ and recalling the normalization of the measure $d\xi_1$. Fix $\xi_1$. From the transversality of $S_1$ and $S_2$ we see that $|T_{\xi_1,i_1} \cap T_{\xi_2,i_2}| \lesssim \delta^3$. Thus we reduce to showing that

$$\sum_{\xi_2} \sum_{i_1} \sum_{i_2} f_1(\xi_1, i_1) f_2(\xi_2, i_2) C_{\xi_1,\xi_2}(i_1, i_2) \lesssim \delta^{-2-\varepsilon} \|f_1(\xi_1, \cdot)\|_{L^2_i} \|f_2\|_{L^2_\xi(L^1_i)}$$

where we define $C_{\xi_1,\xi_2}(i_1, i_2)$ as the characteristic function of the set

$$\{(i_1, i_2) : T_{\xi_1,i_1} \cap T_{\xi_2,i_2} \neq \emptyset\}.$$

(Note that, if we used the bound $C_{\xi_1,\xi_2}(i_1,i_2) \leq 1$ at this point, we would recover the estimate $K^*_{S_1,S_2}((1,1) \times (1,1) \to 1)$.) By Cauchy-Schwarz in the $i_1$ index, it suffices to show that

$$\sum_{i_1} \left( \sum_{\xi_2} \sum_{i_2} f_2(\xi_2, i_2) C_{\xi_1,\xi_2}(i_1,i_2) \right)^2 \lesssim \delta^{-4-\varepsilon} \|f_2\|^2_{L^2_\xi(L^1_i)} \tag{12}$$

We write the left-hand side of (12) as

$$\sum_{\xi_2} \sum_{i_2} f_2(\xi_2, i_2) \sum_{\xi_2'} \sum_{i_2'} f_2(\xi_2', i_2') \sum_{i_1} C_{\xi_1,\xi_2}(i_1,i_2) C_{\xi_1,\xi_2'}(i_1,i_2').$$

We now apply a simple geometric observation.

Figure 3. Three tubes $T_{\xi_2,i_2}$, $T_{\xi_2',i_2'}$, $T_{\xi_1,i_1}$ before and after the projection $\pi$, and the intersections which contribute to $\sum_{i_1} C_{\xi_1,\xi_2}(i_1,i_2) C_{\xi_1,\xi_2'}(i_1,i_2')$.

**Lemma 4.2.** *For $\xi_1$, $\xi_2$, $\xi_2'$, $i_2$ and $i_2'$, fixed, we have*

$$\sum_{i_1} C_{\xi_1,\xi_2}(i_1,i_2) C_{\xi_1,\xi_2'}(i_1,i_2') \lesssim \frac{1}{\delta + \angle(n_2(\xi_2), n_2(\xi_2'))}.$$

**Proof** Fix $\xi_1$, and let $\pi$ be the orthogonal projection of $\mathbf{R}^3$ onto $T_{\xi_1} S_1$. Observe that $C_{\xi_1,\xi_2}(i_1,i_2)$ is only non-vanishing when the disk $\pi(T_{\xi_1,i_1})$ is in a dilate of $\pi(T_{\xi_2,i_2})$. From (9) and elementary geometry we see that $\pi(T_{\xi_2,i_2})$ and $\pi(T_{\xi_2',i_2})$ at most intersect in a rectangle of size $C\delta \times C\delta[\delta + \angle(n_2(\xi_2), n_2(\xi_2'))]^{-1}$, and the claim follows. See Figure 3. ■

Thus (12) reduces to

$$\sum_{\xi_2}\sum_{i_2} f_2(\xi_2,i_2)\sum_{\xi_2'}\sum_{i_2'} f_2(\xi_2',i_2')\frac{1}{\delta+\angle(n_2(\xi_2),n_2(\xi_2'))}\lesssim \delta^{-4-\varepsilon}\|f_2\|^2_{L^2_\xi(L^1_i)}$$

Write $F(\xi_2)=\|f_2(\xi_2,\cdot)\|_{L^1_i}$. If we simplify the above estimate using $F$ and the definition of the measure $d\xi$, we reduce to

$$\sum_{\xi_2}\sum_{\xi_2'} F(\xi_2)F(\xi_2')\frac{1}{\delta+\angle(n_2(\xi_2),n_2(\xi_2'))}\lesssim \delta^{-2-\varepsilon}(\sum_{\xi_2}|F(\xi_2)|^2)^{1/2}(\sum_{\xi_2'}|F(\xi_2')|^2)^{1/2}.$$

By Schur's test, this will follow if we can show

$$\sum_{\xi_2'}\frac{1}{\delta+\angle(n_2(\xi_2),n_2(\xi_2'))}\lesssim \delta^{-2-\varepsilon}$$

uniformly in $\xi_2$, and similarly with $\xi_2$, $\xi_2'$ reversed. By symmetry it suffices to show the displayed estimate. By a dyadic decomposition it suffices to show that

$$\#\{\xi_2':\angle(n_2(\xi_2),n_2(\xi_2'))\sim 2^{-k}\}\sim \delta^{-2}2^{-k}$$

for all $\delta\lesssim 2^{-k}\lesssim 1$. But this is a consequence of the hypothesis that $S_2$ always has exactly one vanishing principal curvature. ■

By interpolation between the trivial estimate $K^*_{S_1,S_2}((1,\infty)\times(1,\infty)\to\infty)$, $K^*_{S_1,S_2}((1,1)\times(1,1)\to 1)$ and $K^*_{S_1,S_2}((4/3,4/3)\times(4/3,4/3)\to 1)$, we prove that $K^*_{S_1,S_2}((p,r)\times(p,r)\to q)$ holds for $1\le q\le\infty$, $r\le 4q/3$ and $\frac{1}{q'}+\frac{1}{r}\ge\frac{1}{p}$. In the rest of this section we indicate briefly why this is the sharp result.

First consider a one–sheeted hyperboloid

$$H=\{x_3^2=x_1^2+x_2^2-1\}.$$

This is a doubly ruled surface, made up of straight lines which are normal to the cone $x_3^2=x_1^2+x_2^2$. We set

$$S_1=\{x_3^2=x_1^2+x_2^2, 0\le\arg(x,y)\le\pi/2\}$$

and

$$S_2=\{x_3^2=x_1^2+x_2^2,\pi\le\arg(x,y)\le 3\pi/2\}.$$

For each $\xi\in\mathcal{E}_t$ we can pick $i=i(\xi)$ so that $T^\delta_{\xi,i(\xi)}$ is contained in a $\delta$–neighborhood of $H$. We also define $f_t(\xi,i)$ as the characteristic function of the set $\{(\xi,i):\ i=i(\xi)\}$, so that $\Pi_t X_t^* f_t(\xi,i)\ge 1/\delta^2$ in a $\delta$-neighborhood of a big part of $H$. Hence, $\|\Pi_t X_t^* f_t\|_q\ge\delta^{1/q-2}$, while $\|f_t\|_{L^p_\xi(L^q_i)}=1$. Hence, we deduce $q\ge 1$.

To show that $r\le 4q/3$ take $f_t(\xi,i)=1$ for all $\xi$ and all $i$. Then $\Pi_t X_t^* f_t\ge\delta^{-4}$ on a big part of the unit ball, which is enough for our purposes.

Finally, take a plane spanned by two light rays and denote by $P$ a $\delta$-neighborhood of that plane. Denote $\tilde{\mathcal{E}}_t$ the subset of $\mathcal{E}_t$ so that $n_t(\xi)$ is contained in that plane for all $\xi\in\tilde{\mathcal{E}}_t$. Take $f_t$ the characteristic function of $\{(\xi,i):\ T^\delta_{\xi,i}\subset P\}$. This example shows the necessity of the last condition, $1/p\le 1/q'+1/r$.

## 5. Proof of Lemma 2.3

This section will be a variant of the arguments in [3] (see also [7], [12], [13], [16], [17]).

Let $\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{E}_2$ be as in Section 1 with $\delta = \frac{1}{R}$. Using the circle of ideas first developed by Fefferman and Córdoba, we partition $S_t$ into caps $C_\xi$, for $\xi \in \mathcal{E}_t$, defined by

$$C_\xi = S_t \cap B(\xi, R^{-1}).$$

By the smoothness of $S_t$ and the hypothesis $R \gg 1$ we see that the $C_\xi$ are essentially disks of diameter $1/R$ oriented in the direction $n_t(\xi)$, which form a finitely overlapping cover of $S_t$. Without loss of generality, we can assume that there is a disjoint family of caps $\{C_\xi\}_{\xi \in \tilde{\mathcal{E}}_t}$ such that the support of $f_t$ is contained in $\cup_{\xi \in \tilde{\mathcal{E}}_t} C_\xi$. We decompose

$$f_t = \sum_{\xi \in \tilde{\mathcal{E}}_1} f_{\xi,t}$$

where the $f_{\xi,t}$ are the restrictions of $f_t$ to $C_\xi$. Informally speaking, the restriction hypothesis $R^*_{S_1,S_2}(\frac{2}{r} \times \frac{2}{r} \to q, \alpha)$ implies a version of the localized "square-function" estimate on $R^2$-balls

$$\|\prod_{t=1}^2 \sum_{\xi \in \tilde{\mathcal{E}}_t} \widehat{f_{\xi,t} d\sigma_t}\|_{L^q(B(0,R^2))} \lesssim R^\alpha R^{(n-1)(2-r)-\frac{n}{q}} \|\prod_{t=1}^2 (\sum_{\xi \in \tilde{\mathcal{E}}_t} |\widehat{f_{\xi,t} d\sigma_t}|^{2/r})^{\frac{r}{2}}\|_{L^q(B(0,R^2))}, \tag{13}$$

while the Kakeya hypothesis $K^*_{S_1,S_2}((\frac{pr}{2}, r) \times (\frac{pr}{2}, r) \to \frac{qr}{2})$ controls the square function on $R^2$-balls:

$$\|\prod_{t=1}^2 (\sum_{\xi \in \tilde{\mathcal{E}}_t} |\widehat{g_{\xi,t} d\sigma_t}|^{2/r})^{\frac{r}{2}}\|_{L^q(B(0,R^2))} \lesssim R^{\frac{n}{q}-(n-1)(2-r)+\varepsilon} \prod_{t=1}^2 \|g_t\|_p. \tag{14}$$

Combining (13) with (14), we obtain the desired conclusion $R^*_{S_1,S_2}(p \times p \to q, \alpha/2 + \varepsilon)$.

We now begin the rigorous proof of Lemma 2.3. The reader is advised not to take the various powers of $R$ in the following argument too seriously; however, we remark that these powers of $R$ are natural in the sense that every estimate in the following is sharp when $f_t \equiv 1$ and $\alpha = 0$, in which case $\widehat{f_1 d\sigma_1}$, $\widehat{f_2 d\sigma_2}$ are comparable to 1 within $O(1)$ of the origin.

We will not prove (13) exactly, but a version which suffices for our purposes. To do so, we need a discrete version of $R^*_{S_1,S_2}(\frac{2}{r} \times \frac{2}{r} \to q, \alpha)$, namely

**Lemma 5.1.** *For $\{a_{\xi,t}\} \subset \mathbf{C}$ we have*

$$\|\prod_{t=1}^2 \sum_{\xi \in \tilde{\mathcal{E}}_t} a_{\xi,t} e^{2\pi i \xi \cdot x}\|_{L^q_x(B(0,R))} \lesssim R^\alpha R^{(n-1)(2-r)} \prod_{t=1}^2 (\sum_{\xi \in \tilde{\mathcal{E}}_t} |a_{\xi,t}|^{2/r})^{\frac{r}{2}}.$$

**Proof** We write

$$\sum_{\xi\in\tilde{\mathcal{E}}_t} a_{\xi,t}e^{2\pi i\xi\cdot x} = C\sum_{\xi\in\tilde{\mathcal{E}}_t}\int_{C_\xi} a_{\xi,t}e^{2\pi i\xi\cdot x}\,d\sigma(\eta)R^{n-1}$$

and

$$\int_{C_\xi} a_{\xi,t}e^{2\pi i\xi\cdot x}\,d\sigma(\xi) = \int_{C_\xi} a_{\xi,t}e^{2\pi i(\xi-\eta)\cdot x}e^{2\pi i\eta\cdot x}\,d\sigma(\eta).$$

We use the Taylor expansion of the exponential

$$e^{2\pi i(\xi-\eta)\cdot x} = \sum_{k=0}^{\infty}\frac{(2\pi i(\xi-\eta)\cdot x)^k}{k!} = \sum_\gamma c_\gamma(\xi-\eta)^\gamma x^\gamma,$$

where $\gamma$ denotes a multiindex and the coefficients $c_\gamma$ are decreasing faster than any exponential when $|\gamma|\to\infty$. Thus,

$$\sum_{\xi\in\tilde{\mathcal{E}}_t} a_{\xi,t}e^{2\pi i\xi\cdot x} = \sum_\gamma c_\gamma\sum_{\xi\in\tilde{\mathcal{E}}_t}\int_{C_\xi} a_{\xi,t}R^{n-1}(R(\xi-\eta))^\gamma e^{2\pi i\eta\cdot x}\,d\sigma(\eta)\left(\frac{x}{R}\right)^\gamma.$$

Then

$$|\sum_{\xi\in\tilde{\mathcal{E}}_t} a_{\xi,t}e^{2\pi i\xi\cdot x}| = |\sum_\gamma c_\gamma\widehat{\phi_\gamma\,d\sigma}(x)\left(\frac{x}{R}\right)^\gamma| \lesssim \sum_\gamma |c_\gamma||\widehat{\phi_\gamma\,d\sigma}(x)|$$

where

$$\phi_\gamma(\eta) = \sum_{\xi\in\tilde{\mathcal{E}}_t}\chi_{C_\xi}a_{\xi,t}R^{n-1}(R(\xi-\eta))^\gamma.$$

To end the proof of the lemma, we just have to apply $R^*_{S_1,S_2}(\frac{2}{r}\times\frac{2}{r}\to q,\alpha)$ and use the decay of $c_\gamma$ and the fact that we have the pointwise estimate $|\phi_\gamma|\leq R^{n-1}\sum_{\xi\in\tilde{\mathcal{E}}_t}a_{\xi,t}\chi_{C_\xi}$. ■

With this lemma, we are ready to prove a version of (13). Fix $z\in B(0,R^2)$. For $x$, $y\in B(z,R)$ we have

$$\sum_{\xi\in\tilde{\mathcal{E}}_t}\widehat{f_{\xi,t}d\sigma_t}(x)\sum_{\xi\in\tilde{\mathcal{E}}_t}\int_{C_\xi} f_t(\eta)e^{2\pi i(\xi-\eta)(x-y)}e^{-2\pi i\eta y}\,d\sigma_t(\eta)e^{-2\pi i\xi(x-y)}.$$

As before, we Taylor expand

$$e^{2\pi i(\xi-\eta)(x-y)}\sum_\gamma c_\gamma(\xi-\eta)^\gamma(x-y)^\gamma.$$

We define $f_{\gamma,t}(\eta)=\sum_{\xi\in\tilde{\mathcal{E}}_t}f_{\xi,t}(\eta)(R(\xi-\eta))^\gamma\chi_{\tilde{C}_\xi}(\eta)$ and $f_{\gamma,\xi,t}(\eta)=f_{\xi,t}(\eta)(R(\xi-\eta))^\gamma$, obtaining

$$\sum_{\xi\in\tilde{\mathcal{E}}_t}\widehat{f_{\xi,t}d\sigma_t}(x)\sum_\gamma c_\gamma\Big[\sum_{\xi\in\tilde{\mathcal{E}}_t}\widehat{f_{\gamma,\xi,t}d\sigma_t}(y)e^{2\pi i\xi(x-y)}\Big]\left(\frac{x-y}{R}\right)^\gamma.$$

Therefore,

$$\|\prod_{t=1}^{2}\sum_{\xi\in\tilde{\mathcal{E}}_t}\widehat{f_{\xi,t}d\sigma_t}\|_{L^q(B(z,R))}$$
$$\lesssim \sum_{\gamma_1}2^{-N|\gamma_1|}\sum_{\gamma_2}2^{-N|\gamma_2|}\left\|\prod_{t=1}^{2}\sum_{\xi\in\tilde{\mathcal{E}}_t}\widehat{f_{\gamma_t,\xi,t}d\sigma_t}(y)e^{2\pi i\xi(\cdot-y)}\right\|_{L^q(B(z,R))}$$

where $N$ is arbitrarily large. We use the lemma to estimate this by

$$\sum_{\gamma_1}2^{-N|\gamma_1|}\sum_{\gamma_2}2^{-N|\gamma_2|}R^{\alpha}R^{(n-1)(2-r)}\prod_{t=1}^{2}(\sum_{\xi\in\tilde{\mathcal{E}}_t}|\widehat{f_{\gamma_t,\xi,t}d\sigma_t}(y)|^{2/r})^{\frac{r}{2}}.$$

Averaging this in $L^q$ with respect to $y\in B(z,R)$ we obtain

$$\|\prod_{t=1}^{2}\sum_{\xi\in\tilde{\mathcal{E}}_t}\widehat{f_{\xi,t}d\sigma_t}\|_{L^q(B(z,R))}$$
$$\lesssim \|\sum_{\gamma_1}2^{-N|\gamma_1|}\sum_{\gamma_2}2^{-N|\gamma_2|}R^{\alpha}R^{(n-1)(2-r)}R^{-n/q}\prod_{t=1}^{2}(\sum_{\xi\in\tilde{\mathcal{E}}_t}|\widehat{f_{\gamma_t,\xi,t}d\sigma_t}|^{2/r})^{\frac{r}{2}}\|_{L^q(B(z,R))}.$$

Finally, we take a $R$-separated set of points $z\in B(0,R^2)$. Raising the last inequality to $q$ and summing on $z$ we obtain

$$\|\prod_{t=1}^{2}\sum_{\xi\in\tilde{\mathcal{E}}_t}\widehat{f_{\xi,t}d\sigma_t}\|_{L^q(B(0,R^2))}$$
$$\lesssim\|\sum_{\gamma_1}2^{-N|\gamma_1|}\sum_{\gamma_2}2^{-N|\gamma_2|}R^{\alpha}R^{(n-1)(2-r)}R^{-n/q}\prod_{t=1}^{2}(\sum_{\xi\in\tilde{\mathcal{E}}_t}|\widehat{f_{\gamma_t,\xi,t}d\sigma_t}|^{2/r})^{\frac{r}{2}}\|_{L^q(B(0,R^2))}.$$

By the triangle inequality, we majorize this by

$$\sum_{\gamma_1}2^{-N|\gamma_1|}\sum_{\gamma_2}2^{-N|\gamma_2|}R^{\alpha}R^{(n-1)(2-r)}R^{-n/q}\|\prod_{t=1}^{2}(\sum_{\xi\in\tilde{\mathcal{E}}_t}|\widehat{f_{\gamma_t,\xi,t}d\sigma_t}|^{2/r})^{\frac{r}{2}}\|_{L^q(B(0,R^2))},$$

which is a variant of (13).

Note that $|f_{\gamma_t,t}(\eta)|\le\sum_{\xi\in\tilde{\mathcal{E}}_t}|f_{\gamma,\xi,t}(\eta)|\chi_{C_\xi}(\eta)\le C|f_t(\eta)|$. Hence, the theorem will follow from (14).

We want to estimate

$$\int_{B(0,R^2)}\prod_{t=1}^{2}(\sum_{\xi\in\tilde{\mathcal{E}}_t}|\widehat{g_{\xi,t}d\sigma_t}(x)|^{2/r})^{\frac{rq}{2}} \tag{15}$$

Let $\psi_\xi$ be a Schwarz function which is comparable to 1 on $C_\xi$ and rapidly decreasing away from this cap, and whose Fourier transform satisfies the pointwise estimate

$$|\hat{\psi}_\xi(x)|\lesssim R^{-n-1}\chi_{R^2\tilde{T}_0^\xi}(x),$$

where $\tilde{T}_0^\xi$ is a thickening of $T_0^\xi$, and $R^2\tilde{T}_0^\xi = \{R^2x : x \in \tilde{T}_0^\xi\}$.

If we define $\tilde{g}_{\xi,t} = g_{\xi,t}/\psi_\xi$, we have the estimate

$$|\widehat{g_{\xi,t}\,d\sigma_t}(x)||\widehat{\tilde{g}_{\xi,t}\,d\sigma_t} * \widehat{\psi_\xi}(x)| \lesssim R^{-n-1}\int_{x+R^2\tilde{T}_0^\xi} |\widehat{\tilde{g}_{\xi,t}\,d\sigma_t}(y)|\ dy.$$

From Hölder's inequality and (4) we thus obtain

$$\begin{aligned}|\widehat{g_{\xi,t}\,d\sigma_t}(x)|^{2/r} &\lesssim (R^{-n-1}\int_{x+R^2\tilde{T}_0^\xi} |\widehat{\tilde{g}_{\xi,t}\,d\sigma_t}(y)|^2\ dy)^{1/r}\\ &\lesssim X^*_{\frac{1}{R},S_t}G_{\xi,t}(\frac{x}{R^2})\end{aligned}$$

where

$$G_{\xi,t}(\xi', i) = \delta_{\xi,\xi'}(R^{n-1}\int_{\tilde{T}_i^\xi} |\widehat{\tilde{g}_{\xi,t}\,d\sigma_t}(R^2x)|^2\ dx)^{1/r}$$

and $\delta_{\xi,\xi'}$ is the Kronecker delta.

From this we see that (15) is majorized by

$$\int_{B(0,R^2)} \prod_{t=1}^2 (\sum_{\xi\in\tilde{\mathcal{E}}_t} X^*_{\frac{1}{R},S_t}G_{\xi,t}(\frac{x}{R^2}))^{qr/2}\ dx.$$

We rescale $x$ by $R^2$ and simplify this as

$$R^{2n}\int_{B(0,1)} \prod_{t=1}^2 X^*_{\frac{1}{R},S_t}G_t(x)^{qr/2}\ dx, \tag{16}$$

where

$$G_t(\xi, i) = \sum_{\xi'} G_{\xi',t}(\xi, i)(R^{n-1}\int_{\tilde{T}_i^\xi} |\widehat{\tilde{g}_{\xi,t}}(R^2x)|^2\ dx)^{1/r}.$$

Since (16) majorizes (15), it remains to show that

$$R^{n+(n-1)(2-r)q}\int_{B(0,1)} \prod_{t=1}^2 (X^*_{\frac{1}{R},S_t}G_t(x))^{qr/2}\ dx \lesssim R^\varepsilon \prod_{t=1}^2 (\|g_t\|_p)^q,$$

which we raise to the power $2/qr$ as

$$R^{\frac{2n}{qr}+2(\frac{2}{r}-1)(n-1)}\|\prod_{t=1}^2 X^*_{\frac{1}{R},S_t}G_t\|_{qr/2} \lesssim R^\varepsilon \prod_{t=1}^2 (\|g_t\|_p)^{2/r}. \tag{17}$$

On the other hand, from the definition of the hypothesis $K^*_{S_1,S_2}((\frac{pr}{2}, r)\times(\frac{pr}{2}, r) \to \frac{qr}{2})$ we have

$$\|\prod_{t=1}^2 X^*_{\frac{1}{R},S_t}G_t\|_{qr/2} \lesssim R^{2n-\frac{2n}{qr}-2+\varepsilon}\prod_{t=1}^2 \|G\|_{L_\xi^{pr/2}L_i^r}$$

Comparing this with (17), we see that we will be done once we show that

$$R^{\frac{2n}{qr}+2(\frac{2}{r}-1)(n-1)}R^{2n-\frac{2n}{qr}-2}\prod_{t=1}^2 \|G_t\|_{L_\xi^{pr/2}L_i^r} \lesssim \prod_{t=1}^2 (\|g_t\|_p)^{2/r}.$$

After some algebraic manipulation we see that it suffices to show that

$$R^{2n-2}\|G_t\|^r_{L^{pr/2}_\xi L^r_i} \lesssim \|g_t\|_p^2$$

for $t = 1, 2$.

From the definition of $g_{\xi,t}$ and the measure $d\xi$ we have

$$\|g_t\|_p^2 \sim R^{\frac{2(n-1)}{p}} \|(\|g_{\xi,t}\|_p)^{2/r}\|^r_{L^{pr/2}_\xi},$$

and so it suffices to show that

$$R^{2n-2}\|G_t(\xi, \cdot)\|^r_{L^r_i} \lesssim R^{\frac{2(n-1)}{p}} \|g_{\xi,t}\|_p^2$$

uniformly in $\xi$.

From Hölder's inequality, the hypothesis $p \geq 2$ and the support conditions on $g_\xi$ we have

$$\|g_{\xi,t}\|_2 \lesssim R^{-(n-1)(\frac{1}{2}-\frac{1}{p})}\|g_{\xi,t}\|_p,$$

and so after some algebra we reduce ourselves to

$$\|G_t(\xi, \cdot)\|^r_{L^r_i} \lesssim R^{-n-1}\|g_{\xi,t}\|_2^2.$$

However, the left-hand side is majorized by

$$R^{n-1}\int_{B(0,C)} |\widehat{\tilde{g}_{\xi,t}\, d\sigma_t}(R^2 x)|^2 \; dx \lesssim R^{-n-1}\|\widehat{\tilde{g}_{\xi,t}\, d\sigma_t}\|^2_{L^2(B(0,R^2))},$$

and the claim follows from the well known estimate $R^*(2 \to 2, 1/2)$ (see e.g. [11], [12]) and the pointwise comparability of $g_{\xi,t}$ and $\tilde{g}_{\xi,t}$. ∎

## 6. Proof of Lemma 2.4

This proof is essentially the one in [6]. The argument works in general dimension $\mathbf{R}^n$.

We begin the proof of Lemma 2.4 by some interpolation theory and duality. Assume that $R^*_{S_1,S_2}(2 \times 2 \to q, \alpha)$ holds, and let $p$ be as in the statement of the lemma.

To prove the conclusion $R^*_{S_1,S_2}(2 \times 2 \to p + \varepsilon)$ of the lemma, it suffices to prove the weak-type estimate

$$|\{\prod_{t=1}^{2}|\widehat{f_t d\sigma_t}| > \lambda\}| \lesssim \lambda^{-p} \tag{18}$$

for all $1 \gtrsim \lambda > 0$ and all $f_t$, $t = 1, 2$ such that $\|f_t\|_2 \sim 1$. This is because the estimate $R^*_{S_1,S_2}(2 \times 2 \to \infty)$ (for instance) is trivially true.

One may replace (18) with the equivalent

$$|E| \lesssim \lambda^{-p},$$

where $E = \{\mathrm{Re}\prod_{t=1}^{2} \widehat{f_t d\sigma_t} > \lambda\}$. We may, of course, assume $|E| \gtrsim 1$. By construction we have

$$\lambda |E| \lesssim \|\chi_E \prod_{t=1}^{2} \widehat{f_t d\sigma_t}\|_1$$

so it thus suffices to show that

$$\|\chi_E \prod_{t=1}^{2} \widehat{f_t d\sigma_t}\|_1 \lesssim |E|^{1/p'}.$$

Fix $\lambda$, $f_1$, $f_2$, so that $E$ is also fixed. It suffices to show that

$$\|\chi_E \widehat{g_1 d\sigma_1}\widehat{g_2 d\sigma_2}\|_1 \lesssim |E|^{1/p'} \|g_1\|_2 \|g_2\|_2 \tag{19}$$

for arbitrary $L^2$ functions $g_1$, $g_2$; in order to apply some duality arguments it will be important that $g_1$, $g_2$ are completely independent of $f_1$, $f_2$.

Fix $g_2$ with $\|g_2\|_2 \sim 1$. It suffices to show that

$$\|Tg_1\|_1 \lesssim |E|^{1/p'} \|g_1\|_2,$$

where $T = T_{E,g_2}$ is the linear operator

$$Tg_1 = \chi_E \widehat{g_1 d\sigma_1}\widehat{g_2 d\sigma_2}.$$

By duality, it suffices to show that

$$\|T^* F\|_{L^2(d\sigma_1)} \lesssim |E|^{1/p'} \|F\|_\infty$$

where $T^*$ is the adjoint operator

$$T^* F = \mathcal{F}^{-1}(\chi_E \widehat{g_2 d\sigma_2} F),$$

and $\mathcal{F}^{-1}$ is the inverse Fourier transform. We may assume that $\|F\|_\infty \lesssim 1$.

By squaring this and applying Plancherel's theorem, we reduce ourselves to showing that

$$|\langle \tilde{F} * \widehat{d\sigma_1}, \tilde{F}\rangle| \lesssim |E|^{2/p'}, \tag{20}$$

where $\tilde{F} = \chi_E \widehat{g_2 d\sigma_2} F$. Note that the hypotheses on $F$, $g_2$ and the Tomas–Stein theorem $R^*(2 \to \frac{2\sigma+2}{\sigma})$ imply

$$\|\tilde{F}\|_{L^1} \leq \|\chi_E\|_{\frac{2\sigma+2}{\sigma+2}} \|\widehat{g_2\, d\sigma_2}\|_{\frac{2\sigma+2}{\sigma}} \|F\|_\infty \lesssim |E|^{\frac{\sigma+2}{2\sigma+2}}. \tag{21}$$

Let $R > 1$ be a quantity to be chosen later. Let $\phi$ be a bump function which equals 1 on $|x| \lesssim 1$ and vanishes for $|x| \gg 1$, and write $d\sigma_1 = d\sigma_1^R + d\sigma_{1\,R}$, where

$$\widehat{d\sigma_{1\,R}}(x) = \phi\left(\frac{x}{R}\right)\widehat{d\sigma_1}(x). \tag{22}$$

From (6) we have

$$\|\widehat{d\sigma_1^R}\|_\infty \lesssim R^{-\sigma},$$

and so by (21) we have

$$|\langle \tilde{F} * \widehat{d\sigma_1^R}, \tilde{F}\rangle| \lesssim R^{-\sigma} |E|^{\frac{\sigma+2}{\sigma+1}}.$$

We now choose $R$ to be

$$R = |E|^{\frac{1}{\sigma}(\frac{\sigma+2}{\sigma+1} - \frac{2}{p'})} \tag{23}$$

so that the contribution of $d\sigma_1^R$ to (20) is acceptable. Thus (20) reduces to

$$|\langle \tilde F * \widehat{d\sigma_{1\,R}}, \tilde F\rangle| \lesssim |E|^{2/p'}. \tag{24}$$

By Plancherel's theorem, we have

$$|\langle \tilde F * \widehat{d\sigma_{1\,R}}, \tilde F\rangle| \lesssim \int |\hat{\tilde F}|^2\ |d\sigma_{1\,R}|.$$

From (22), Plancherel's theorem, and the rapid decay of $\hat\phi$ we have the estimate

$$|d\sigma_{1\,R}| \lesssim R(1 + R\mathrm{dist}(\xi, S_1))^{-N} d\xi$$

for some large $N$. Decomposing this dyadically, we see that

$$|\langle \tilde F * \widehat{d\sigma_{1\,R}}, \tilde F\rangle| \lesssim \sum_{j=0}^\infty R 2^{-jN} \|\hat{\tilde F}\|^2_{L^2(S_{1,2^j R})},$$

where $S_{1,2^{-j}R}$ is the $2^j R^{-1}$ neighbourhood of $S_1$. We treat the $j=0$ case; the other cases are similar but are aided by the factor $2^{-jN}$. To control this contribution to (24), it suffices to show

$$\|\hat{\tilde F}\|_{L^2(S_{1,R})} \lesssim R^{-1/2}|E|^{1/p'}.$$

From the definition of $\tilde F$, this will follow if we can show

$$\|\mathcal F^{-1}(\chi_E \widehat{g_2 d\sigma_2} F)\|_{L^2(S_{1,R})} \lesssim R^{-1/2}|E|^{1/p'}\|F\|_\infty$$

for all $F$. By duality and the hypothesis on $g_2$, it thus suffices to show

$$\|\chi_E \widehat{\tilde g_1}\widehat{g_2 d\sigma_2}\|_1 \lesssim R^{-1/2}|E|^{1/p'}\|\tilde g_1\|_2\|g_2\|_2$$

for all $\tilde g_1$ supported on the annulus $S_{1,R}$. This estimate should be compared with (19).

We now fix $\tilde g_1$, and allow $g_2$ to vary instead, and apply the above argument again but with $g_2$ playing the role of $g_1$. This allows us to reduce again to

$$\|\chi_E \widehat{\tilde g_1}\widehat{\tilde g_2}\|_1 \lesssim R^{-1/2}R^{-1/2}|E|^{1/p'}\|\tilde g_1\|_2\|\tilde g_2\|_2$$

where for $t = 1, 2$, $\tilde g_t$ is an arbitrary function on the $1/R$ neighbourhood of $S_{t,R}$.

We are finally in a position to apply the hypothesis $R^*(2\times 2 \to q, \alpha)$. By Hölder's inequality it suffices to show

$$\|\widehat{\tilde g_1}\widehat{\tilde g_2}\|_q \lesssim |E|^{-1/q'}R^{-1/2}R^{-1/2}|E|^{1/p'}\|\tilde g_1\|_2\|\tilde g_2\|_2. \tag{25}$$

Let $\psi$ be a bump function whose Fourier transform is positive on the unit ball, and for any $x$ let $\psi_R^x(\xi) = e^{2\pi i x\cdot\xi}\psi(\xi/R)$. Applying $R^*(2\times 2 \to q, \alpha)$ and using Plancherel's theorem, we obtain

$$\|(\widehat{\psi_R^x})^2\widehat{\tilde g_1}\widehat{\tilde g_2}\|_q \lesssim R^\alpha \prod_{t=1}^2 R^{-1/2}\|\widehat{\psi_R^x}\widehat{\tilde g_t}\|_2.$$

If we let $x$ range over a maximal $R$-separated set of $\mathbf{R}^3$ and then sum, using the triangle inequality on the left and the Cauchy-Schwarz inequality on the right, we obtain

$$\|\widehat{\tilde{g}}_1\widehat{\tilde{g}}_2\|_q \lesssim R^\alpha \prod_{t=1}^{2} R^{-1/2}\|\widehat{\tilde{g}}_t\|_2.$$

Comparing this with (25), we see that we will be done if

$$R^\alpha \lesssim |E|^{-1/q'}|E|^{1/p'} = |E|^{\frac{1}{q}-\frac{1}{p}}.$$

But this follows from (23) and the definition $\frac{1}{p}\left(1+\frac{2\alpha}{\sigma}\right) < \frac{1}{q}+\frac{\alpha}{1+\sigma}$. ■

In order for Lemma 2.4 to be useful, one must find a local estimate $R^*_{S_1,S_2}(2\times 2 \to q, \alpha)$ with a very small value of $\alpha$. Since our main tool for lowering the value of $\alpha$ is Lemma 2.3, we will try to use this lemma as many times as possible. This explains the iterative nature of the argument in Section 7.

We compare this lemma with two other results of the same form (that is, converting a local restriction theorem to a global restriction theorem), but which are concered with linear restriction estimates rather than bilinear. All three results are based on the Tomas–Stein argument, although the methods are quite different in other respects.

The first result is an analogue of Lemma 2.4 developed by Bourgain [3, 7], and refined slightly in [12], for the linear problem:

**Lemma 6.1.** [3, 7, 12] *Let $S$ be a surface with decay $\sigma$ in the sense of* (6). *If $p, q, \alpha$ are such that $\sigma + 1 > \alpha q$, then $R^*_S(p \to q, \alpha)$ implies $R^*_S(\tilde{p} \to \tilde{q})$ whenever*

$$\tilde{q} > 2 + \frac{q}{\sigma+1-\alpha q}, \quad \frac{\tilde{q}}{\tilde{p}} < 1 + \frac{q}{p}\frac{1}{\sigma+1-\alpha q}.$$

**Proof** This lemma was essentially proven in [12], which refined the ideas in [3, 7]); we give only a sketch here.

Assume $R^*_S(p \to q, \alpha)$. To prove the lemma it suffices to provethe restricted weak-type endpoint estimate

$$|\{\mathrm{Re}\widehat{\chi_\Omega d\sigma} > \lambda\}| \lesssim \lambda^{-\tilde{q}}|\Omega|^{\tilde{q}/\tilde{p}}$$

for all $\lambda > 0$ and $\Omega \subseteq S$, with

$$\tilde{q} = 2 + \frac{q}{\sigma+1-\alpha q}, \frac{\tilde{q}}{\tilde{p}} = 1 + \frac{q}{p}\frac{1}{\sigma+1-\alpha q}. \tag{26}$$

It will be convenient to normalize $\chi_\Omega$ in $L^2$, so we rewrite this estimate as

$$|\{\mathrm{Re}\widehat{f d\sigma} > \tilde{\lambda}\}| \lesssim (|\Omega|^{1/2}\tilde{\lambda})^{-\tilde{q}}|\Omega|^{\tilde{q}/\tilde{p}}$$

where $\tilde{\lambda} = |\Omega|^{-1/2}\lambda$ and $f = |\Omega|^{-1/2}\chi_\Omega$.

Fix $\lambda$, $\Omega$, and denote the set in the left-hand side by $E$. By (26) and some algebra, the desired estimate reduces to showing that

$$|E| \lesssim \tilde{\lambda}^{-2}\left(\tilde{\lambda}^{-q}|\Omega|^{q(\frac{1}{p}-\frac{1}{2})}\right)^{\frac{1}{\sigma+1-\alpha q}}. \tag{27}$$

We now invoke Proposition 2.2 of [7] (see also [3, 12]). Since $f$ is normalized in $L^2$, the Proposition applies (trivially generalizing to arbitrary decay $\sigma$, and letting $N \to \infty$), and we have

$$|E| \lesssim \tilde{\lambda}^{-2} \left( R + \sum_{\rho > R, \rho \text{ dyadic}} \rho^{-\sigma} \sup_x |E \cap B(x, \rho)| \right), \tag{28}$$

where $R \gg 1$ is a parameter to be optimized later. Note that if one estimated $|E \cap B(x, \rho)|$ by $|E|$ then this reduces to the weak-type Tomas–Stein estimate.

By Chebyshev's inequality and the definition of $E$ we have

$$\tilde{\lambda} |E \cap B(x, \rho)|^{1/q} \lesssim \|\widehat{f d\sigma}\|_{L^q(B(x,\rho))},$$

and so by the hypothesis $R_S^*(p \to q, \alpha)$ and translation invariance we have

$$\tilde{\lambda} |E \cap B(x, \rho)|^{1/q} \lesssim \rho^\alpha \|f\|_p = \rho^\alpha |\Omega|^{\frac{1}{p} - \frac{1}{2}}.$$

Inserting this into (28) we obtain

$$|E| \lesssim \tilde{\lambda}^{-2} \left( R + \sum_{\rho > R, \rho \text{ dyadic}} \rho^{\alpha q - \sigma} \tilde{\lambda}^{-q} |\Omega|^{q(\frac{1}{p} - \frac{1}{2})} \right).$$

Summing this using the hypothesis $\sigma + 1 > \alpha q$, we obtain

$$|E| \lesssim \tilde{\lambda}^{-2} \left( R + R^{\alpha q - \sigma} \lambda^{-q} |\Omega|^{q(\frac{1}{p} - \frac{1}{2})} \right),$$

and (27) follows by choosing

$$R = \left( \tilde{\lambda}^{-q} |\Omega|^{q(\frac{1}{p} - \frac{1}{2})} \right)^{\frac{1}{\sigma + 1 - \alpha q}}.$$

∎

Like Lemma 2.4, Lemma 6.1 also suffers from some inefficiencies, as $\tilde{q}$ and $\tilde{p}$ are considerably worse exponents than $q$ and $p$, even when $\alpha = 0$. On the other hand, it works for all exponents $p$, which is not the case for Lemma 2.4. It may be possible to combine the arguments in both Lemma 2.4 and Lemma 6.1 and obtain a result which is superior to both.

A third lemma of the same type as the preceding two can be found in [16]:

**Lemma 6.2.** [16] *Let $S$ be a surface with a non-zero decay $\sigma > 0$, in the sense of* (6)*. If $p < 2$ and $0 < \alpha \ll 1$, then $R_S^*(p \to p, \alpha)$ implies $R_S^*(p \to q)$ whenever*

$$\frac{1}{q} > \frac{1}{p} + \frac{C_\sigma}{\log \frac{1}{\alpha}}.$$

In [16] this argument was shown for the optimal decay $\sigma = (n-1)/2$, but it is applicable to more general values of $\sigma$. Again, the Tomas–Stein philosophy is used in the proof. This argument is inferior to Lemma 6.1 for most values of $\alpha$, but is superior for very small $\alpha$; in particular, it only loses an epsilon in the limit $\alpha \to 0$. However, the presence of the logarithm and the unspecified constant $C_\sigma$ renders this result unsuitable for finding quantitative global restriction theorems.

## 7. Proof of Theorem 1.2

The proof of this theorem will be an easy consequence of the above lemmas and estimates, and some interpolation.

Let $S_1$ and $S_2$ be a fixed pair of surfaces of disjoint conic type. For simplicity we drop the $S_1$, $S_2$ subscripts in what follows.

If we specialize Lemma 2.3 to the case $r = 2/q$, $p = 2$ and apply Proposition 2.2, we obtain

**Corollary 7.1.** *If $2 > q \geq 3/2$ and $\alpha \geq 0$, then $R^*(q \times q \to q, \alpha)$ implies $R^*(2 \times 2 \to q, \alpha/2 + \varepsilon)$ for all $\varepsilon > 0$.*

Unfortunately the conclusion of this corollary does not have quite the same form as the hypothesis, and in order to iterate this argument we will have to perform a (rather inefficient) interpolation to convert the estimate back into the form of the hypothesis. After enough iterations of this Corollary, the value of $\alpha$ will be very small, and we can then use Lemma 2.4 to remove the $\alpha$ entirely. Our estimates are certainly not sharp, and additional techniques or estimates could surely be used to improve the final result.

Figure 4. Proof of Theorem 1.2.

From Theorem 2.1 we have

$$R^*(\frac{12}{7} \times \frac{12}{7} \to 2, 0). \tag{29}$$

From the well known estimate $R^*_S(2 \to 2, 1/2)$ we have

$$R^*_{S_1,S_2}(2 \times 2 \to 1, 1) \tag{30}$$

Interpolating this with (29) we obtain

$$R^*(\frac{7}{4} \times \frac{7}{4} \to \frac{7}{4}, \frac{1}{7}).$$

By Corollary 7.1 we obtain

$$R^*(2 \times 2 \to \frac{7}{4}, \frac{1}{14} + \varepsilon).$$

Interpolating this with (29) we obtain

$$R^*(\frac{13}{7} \times \frac{13}{7} \to \frac{13}{7}, \frac{1}{26} + \varepsilon).$$

Applying Corollary 7.1 again we obtain

$$R^*(2 \times 2 \to \frac{13}{7}, \frac{1}{52} + \varepsilon).$$

Interpolating this with (29) again, we obtain

$$R^*(\frac{19}{10} \times \frac{19}{10} \to \frac{19}{10}, \frac{1}{76} + \varepsilon).$$

Applying Corollary 7.1 again we obtain

$$R^*(2 \times 2 \to \frac{19}{10}, \frac{1}{152} + \varepsilon).$$

At this point, further iteration of this scheme becomes counterproductive, so we instead apply Lemma 2.4 (with decay $\sigma = 1/2$) to obtain

$$R^*(2 \times 2 \to 2 - \frac{8}{121} + \varepsilon)$$

as desired. ■

We display the above argument in Figure 4, which is an expanded version of Figure 2. The points 1-7 are as in Figure 2. The points marked by squares represent local restriction theorems $R^*(p \times p \to q, \alpha)$ with the displayed value of $\alpha$. The argument starts with the trace lemma $R^*(2\times 2 \to 1, 1)$ (not pictured), and proceeds along the dashed lines, interpolating with Theorem 2.1 and using Corollary 7.1, until $\alpha$ is small enough to use Lemma 2.4 efficiently.

We would like to remark that the best possible bilinear restriction estimate is $R^*(2 \to \frac{5}{3})$. Adapting the "squashed caps" example in [17] section 2.7 (see also [9]), we can show that $R^*(p \times p \to q)$ is only possible when $\frac{5}{2q} + \frac{3}{p} \leq 3$.

However, the one–sheeted hyperboloid example gives $q \geq 5/3$ as a necessary condition. We sketch the argument as follows. Break up the cone into $\delta$-caps as usual, and place a $L^\infty$-normalized smooth function on each cap with a phase so that its Fourier transform is supported in a $\delta^{-1} \times \delta^{-1} \times \delta^{-2}$-tube which lies on the $\delta^{-1}$-neighbourhood of the one-sheeted hyperboloid. Randomization gives the condition.

## 8. A new restriction estimate for graphs of elliptic functions

In this section $S_1$ and $S_2$ are as in the statement of theorem 1.3.

In [21], Theorem 1, the following x-ray estimate was proven:

$$\|Xf\|_{L_e^{10/3} L_x^{10}} \lesssim \|D^\varepsilon f\|_{5/2},$$

where $D^\varepsilon$ is a fractional derivative operator of arbitrarily small order $\varepsilon > 0$, and $X$ is the standard x-ray transform in $\mathbf{R}^3$. (See [21] for details). The dual of this estimate is

$$\|D^{-\varepsilon} X^* F\|_{5/3} \lesssim \|F\|_{L_e^{10/7} L_x^{10/9}},$$

By discretizing this estimate in the usual manner, we obtain

$$\|X^*_{\delta,S_t} f\|_{5/3} \lesssim \delta^{-\frac{1}{5}-\varepsilon} \|f\|_{L_\xi^{10/7} L_i^{10/9}}$$

for $t = 1, 2$. (The exponent $\delta^{-\frac{1}{5}}$ is the natural choice, as the simple counterexample $f(\xi, i) = \chi_{T^\delta_{\xi,i}}(0)$ shows.) Bilinearizing this estimate using Hölder's inequality, we obtain

$$\|\prod_{t=1}^{2} X^*_{\delta,S_t} f_t\|_{5/6} \lesssim \delta^{-\frac{2}{5}-\varepsilon} \prod_{t=1}^{2} \|f_t\|_{L_\xi^{10/7} L_i^{10/9}}$$

which is

$$K^*_{S_1,S_2}((\frac{10}{7}, \frac{10}{9}) \times (\frac{10}{7}, \frac{10}{9}) \to \frac{5}{6}). \tag{31}$$

Henceforth we will discard the $S_1, S_2$ subscripts from all estimates. The estimate (31) is strictly stronger than the bilinear form of Wolff's Kakeya estimate

$$K^*((\frac{10}{7}, 1) \times (\frac{10}{7}, 1) \to \frac{5}{6})$$

but it is not directly comparable with the bilinear Kakeya estimate

$$K^*((\frac{5}{4}, 1) \times (\frac{5}{4}, 1) \to \frac{5}{6})$$

proven in [17].

From [17], Theorem 2.3 (the analogue of Theorem 2.1 in this paper) we have

$$R^*(\frac{12}{7} \times \frac{12}{7} \to 2, 0). \tag{32}$$

Interpolating this with (30) we obtain

$$R^*(\frac{9}{5} \times \frac{9}{5} \to \frac{3}{2}, \frac{1}{3}). \tag{33}$$

By Lemma 2.3 with $p = 18/7$, $q = 3/2$, $r = 10/9$, $\alpha = 1/3$,we may use this estimate and (31) to obtain

$$R^*(\frac{18}{7} \times \frac{18}{7} \to \frac{3}{2}, \frac{1}{6} + \varepsilon). \tag{34}$$

By Hölder's inequality, we may raise the 18/7 index to 3:

$$R^*(3 \times 3 \to \frac{3}{2}, \frac{1}{6} + \varepsilon).$$

Figure 5. Proof of Theorem 1.3.

The exponents $p = 3, q = 3/2$ satisfy (within an epsilon) the criteria of [17] Theorem 2.2, which gives an equivalence between linear and bilinear restriction estimates. Thus we may convert the bilinear estimate back to a linear one:

$$R^*(3 \to 3, \frac{1}{12} + \varepsilon). \tag{35}$$

We may then apply Lemma 6.1 to remove the 1/12 and obtain (1). We remark that if the Kakeya conjecture $K^*(3/2 \to 3/2)$, which is still unsolved at this time of writing, was inserted into Bourgain's original argument in [7] (see also [12]), one would obtain precisely (1). Informally, while Wolff's x-ray estimate is strictly weaker than the full Kakeya conjecture, this can be compensated for by replacing the Tomas–Stein theorem in the arguments of [7] by the superior Theorem 2.1. On the other hand, if one combined the Kakeya conjecture with the arguments in this paper then one improve upon (1) by a modest amount; for instance, one could interpolate (34) with (33) and insert the interpolated estimate and the Kakeya conjecture into Lemma 2.3.

We now turn to (2). Interpolating (34) with (32) we obtain

$$R^*(2 \times 2 \to \frac{7}{4} + \varepsilon, \frac{1}{14}).$$

From Lemma 2.4 we obtain (2). Finally we prove (3). The bilinear form of (1) is

$$R^*(\frac{26}{11} \times \frac{26}{11} \to \frac{13}{7} + \varepsilon).$$

Interpolating this with (2) we obtain as

$$R^*(\frac{58}{27} \times \frac{58}{27} \to \frac{58}{31} + \varepsilon).$$

This estimate satisfies the criterion of [17], Theorem 2.2, and so one can convert this to (3).

We display the above arguments in Figure 5, which displays both linear $R^*(p \to q, \alpha/2)$ and bilinear $R^*(p \times p \to q/2, \alpha)$ restriction estimates, and is a modification of Figure 3 in [17]. The points $1 - 6$ are respectively: the Tomas Stein theorem; Theorem 2.1 for elliptic surfaces; Theorem 4.1 in [17]; (1); (2); and (3). Note how the arguments for (1) and (2) diverge at the estimate (34). It may be possible to improve these estimates by finding an improved version of Lemma 2.4 or Lemma 6.1 which can utilize (34) directly.

## 9. Generalizations

- There is a version of Theorem 1.3 for surfaces with negative curvature. In that case, the notion of unit–separated subsets of $S$ has to be replaced by another one, adapted to the Minkowski metric. To illustrate this, we consider the case of the parabolic hyperboloid $S = \{x_3 = x_1 x_2\}$. Following the proof of Theorem 2.1, we see that the rotational curvature condition reduces to $|x_1 - y_1||x_2 - y_2| \geq C$ for $(x, \Phi(x)) \in S_1$ and $(y, \Phi(y)) \in S_2$. Under this condition, (2) holds. Moreover, a version of [17], Theorem 2.2 allows us to obtain linear estimates from the bilinear ones. This gives (1) and (3) for this surface.
- One can modify the proof of (1) to show that the three-dimensional Bochner-Riesz conjecture for an elliptic surface (e.g. the sphere) is true for all $p > 4 - 2/7$, by following the procedure sketched in [17]; the close analogy between these two problems is also demonstrated in [7], [16]. It is not clear whether the other restriction estimates obtained in this paper have analogues for the Bochner-Riesz problem.
- In four dimensions, one can obtain a non-trivial bound of the form $R^*_{S_1,S_2}(2\times 2 \to p)$ for some $p < 2$, for pairs of surfaces which satisfy a suitable four-dimensional analogue of the disjoint conic type condition. An inspection of the above arguments reveal that the only steps which need to be carefully checked are the restriction estimate in Theorem 2.1, and the Kakeya-type estimate in Proposition 2.2. The restriction estimate generalizes easily, providing that the surfaces $S_1$, $S_2$ are such that the phase function $\phi$ appearing in Section 3 satisfies the rotational curvature condition. The arguments in Section 4 will no longer give the optimal Kakeya estimate, but they will be able to obtain some improvement over the trivial $K^*(1 \times 1 \to 1)$, providing that $S_1$, $S_2$ are such that Lemma 4.1 obtains.

  In five and higher dimensions, the estimates given by the above arguments become inferior to those given by the usual Strichartz' estimate and Hölder's inequality.

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