

# Acknowledgements

I would like to thank my supervisor Dr. Terry Tao, for all his help with the writing of this Thesis, both in Los Angeles at the University of California, Los Angeles and in Sydney at the University of New South Wales. I would also like to thank the various members of the academic staff who have been very supportive over my honours year, and have helped me to undertake part of this year on exchange without any problems.

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# Chapter 1

## Introduction

We will consider certain families of sequences and analyse their distribution modulo 1. To do this we will give certain criteria for the uniform distribution of sequences and use these criteria to discuss the distributions of sequences.

## 1.1 Introduction

In this report we look at sequences of numbers and determine whether their fractional parts are uniformly distributed in the interval  $[0, 1)$ . The fractional part of a number  $\alpha$  (denoted by  $\{\alpha\}$ ) is given by  $\alpha - \lfloor \alpha \rfloor$ . (Note: we will sometimes work modulo an interval  $[0, H)$ . In this case, we will let  $\{x\}_H$  denote the remainder when  $x$  is divided by  $H$ .)

To begin we define what it means for a sequence of numbers to be uniformly distributed modulo  $H$ , where  $H \in \mathbb{R}$ . An infinite sequence of numbers  $(\alpha_i)$  is said to be uniformly distributed modulo  $H$  if and only if the corresponding sequence of  $(\{\alpha_i\}_H)$  is uniformly distributed in the interval  $[0, H)$ . Thus a sequence  $(\alpha_i)$  is uniformly distributed modulo 1 if its fractional parts are uniformly distributed in the unit interval.

Let us now define what it means to be uniformly distributed in an interval. To do this we first define a function  $\phi$  as follows:

Given an infinite sequence of real numbers,  $(\alpha_i)$  define  $\phi_{[m,n]}^{\alpha_i}(I)$  to be the number of  $i$ 's with  $\alpha_i \in I$  and  $m \leq i \leq n$ .

The sequence  $(\alpha_i)$ ,  $A \leq \alpha_i < B$  is uniformly distributed in the interval  $[A, B)$  if and only if for every  $0 \leq a < b \leq 1$ ,

$$\lim_{N \rightarrow \infty} \frac{\phi_{[1,N]}([a, b))}{N} = \frac{b - a}{B - A} \quad (1.1)$$

What (1.1) is saying is that as  $N$  gets very large, the proportion of  $\alpha_i$ 's lying in a given interval must approach the proportional length of the interval. This is consistent with our intuitive understanding of uniform distribution.

Throughout the course of this report we will often write u.d. mod 1 to mean uniformly distributed modulo 1.

We will consider in very limited detail the distribution of pairs of sequences in  $R^2$ . If  $\mathbf{x} = (x_1, x_2)$  then we write the fractional part of  $\mathbf{x}$  as  $\{\mathbf{x}\} = (\{x_1\}, \{x_2\})$ . Let  $(\mathbf{x}_n)$  be a sequence of vectors in  $R^2$ . We say that  $(\mathbf{x}_n)$  is uniformly distributed modulo  $(H_1, H_2)$  in  $R^2$  if

$$\lim_{N \rightarrow \infty} \frac{\phi_{[1,N]}^{\{\mathbf{x}_n\}}([\mathbf{a}, \mathbf{b}))}{N} = \prod_{j=1}^2 \frac{(b_j - a_j)}{H_j} \quad (1.2)$$

for all intervals  $[\mathbf{a}, \mathbf{b}) = [a_1, b_1) \times [a_2, b_2) \in I^2$ .

Specific families of sequences we will consider include  $\sum c_\alpha n^\alpha$ ,  $\sum c_j n^j \log^{\tau_j} n$  and  $n^\alpha \sin n^\beta$ .

A large number of the results for non trigonometric sequences are already known and have been included in this report for completeness. As a general

rule, it is fairly easy to analyse the behaviour of sequences with a small derivative. Sequences with very small derivatives (e.g.  $\log n$ ) tend not to be u.d. as they are too close to the constant function. The fractional parts of these sequences are not changing quickly enough to be spread out uniformly over the unit interval.

Sequences with slightly larger derivatives (e.g.  $n^\alpha$ ,  $\alpha < 1$ ) are also fairly easy to analyse. This is because it is quite easy to change the sum which results from Weyl's Criterion (section 2.1) into an integral and then evaluate this integral.

Sequences with a larger (e.g.  $n^\alpha$ ,  $\alpha > 1$ ) which can't be dealt with by evaluating the resulting integral can often be dealt with using Van Der Corput's Difference Theorem (section 2.1). This theorem essentially lets us replace a sequence with its derivative and analyse its distribution this way. In many cases it will be possible to evaluate the integral associated with one of the higher order derivatives of a sequence and thus form a conclusion about the distribution of that sequence.

Generally speaking sequences with very large derivatives are difficult to analyse, as the fractional part of the terms in the sequences are varying rapidly. In this report we will discuss the distribution of sequences of the form  $(n^\alpha \sin n^\beta)$ . These trigonometric functions at times have an extremely fast derivative and so can be difficult to analyse. There has previously been little analysis of the distributions of trigonometric sequences. In [7] we find some analysis of the distributions of families of sequences of the form  $f(n) \sin nx$  however this analysis is restricted to probabilistic arguments and only works for almost every value of  $x$ . The methods used in [?] cannot be used to show the uniform distribution of such a sequence for any particular value of  $x$ . An analysis of sequences of the form  $P(n)f(n\alpha)$  where  $P$  is a polynomial with integer powers,  $f$  is a periodic function and  $\alpha$  is a constant can be found in [2]. The analysis encompasses functions of the form  $P(n) \sin n$  however the machinery used is extremely complex. As far as we know these are the only discussions of such families of sequences. In this report an attempt is made to give a general understanding of the behaviour of some trigonometric sequences using less complicated machinery.

We will make a complete analysis of the distribution of sequences of the form  $(n^\alpha \sin n^\beta)$  for  $\beta < 1$  and  $\alpha \in \mathbb{R}$ . The results are contained in Theorems 4.1.2, 4.2.1, 4.3.1 and 4.4.9.

The restriction  $\beta < 1$  slows down the derivative of the sequence as  $n$  increases and thus makes it easier to analyse the distribution.

A partial analysis is made of the distribution of  $n^\alpha \sin n$  which includes the case when  $\alpha < \frac{1}{2}$  and is found in Theorem 4.5.1. In this case we actually discuss the distribution of  $n^\alpha \sin 2\pi an$  where  $a$  is a real number less than 1.

We write  $a = \frac{p}{q} + \delta$  using a theorem due to Dirichlet, replace  $n$  by  $mq + r$  and treat  $n^\alpha \sin 2\pi an$  as a function of  $m$  instead. We then show that the extra factor of  $q\delta$  obtained in the derivative of the sine term will be small in size and so makes the sequence easier to analyse as it now has a (relatively) small derivative.

As an appendix we make a partial analysis of the sequence  $n \sin n$  using the methods found in [2]. Lack of time has meant that a full analysis of this sequence could not be made.

As far as we know, no analysis has been made for sequences with  $\beta > 1$ . This case is very difficult to deal with, as the sine term is oscillating more rapidly as  $n$  increases and so the derivative of the sequence gets very large.

Limited analysis has been made of sequences of the form  $(\alpha^n)$ . Understanding the distribution of this family of sequences would be equivalent to answering the question about whether the digits of  $\pi$  are uniformly distributed. As the derivative of  $\alpha^n$  is extremely large this is a particularly hard problem and will not be addressed in this report.

## 1.2 Notation

Throughout the report  $K$  will be used to represent a constant. The value of  $K$  may vary from line to line.

For a function  $f$  and a set of intervals  $A = \cup_{i=1}^m [a_i, b_i]$  the notation  $[f(x)]_A$  will be used to represent  $f$  evaluated over the intervals in  $A$ . i.e.

$$\int_A f' = [f(x)]_A = \sum_{i=1}^m [f(x)]_{a_i}^{b_i} = \sum_{i=1}^m (f(b_i) - f(a_i))$$

When writing  $\phi_{[m,n]}^{\alpha_i}(I)$  we will often make the following abbreviations.

1. If it is obvious which sequence we are referring to, we will drop the superscript from  $\phi$ .
2. If  $m=1$  we will often not include it in the subscript of  $\phi$ . For example  $\phi_{[1,N]}^{\alpha_i}(I) = \phi_N^{\alpha_i}(I)$
3. If  $I$  is composed of a single interval we will often not include the brackets around the interval. For example,  $\phi_N([a, b])$  will be written as  $\phi_N(a, b)$ .

## Chapter 2

# Some Basic Theorems and Useful Tools

This chapter will be devoted to outlining some of the theorems and tools which will be used throughout the report. As a general rule it is both difficult and tedious to evaluate the function  $\phi_N^{x_n}(a, b)$  for a sequence  $(x_n)$ . For this reason we look for other methods which can be used to determine the distribution of sequences. Many of the proofs in this chapter are standard proofs which have been taken from Chapters 2 and 3 of [6].



## 2.1 Theorems

### 2.1.1 Weyl's Criterion

Weyl's theorems provide necessary and sufficient conditions for sequences of numbers to be uniformly distributed either in the unit interval or modulo 1 based on their exponential sums.

**Theorem 2.1.1** *If  $(\alpha_n)$  is a sequence with  $0 \leq \alpha_n \leq 1$  for  $n = 1, 2, \dots$  then a necessary and sufficient condition for  $(\alpha_n)$  to be uniformly distributed in the unit interval is that*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\alpha_n) = \int_0^1 f(x) dx \quad (2.1)$$

*for every function  $f$  which is Riemann integrable in  $0 \leq x \leq 1$ .*

**Theorem 2.1.2 (Weyl's Criterion)** *If  $(\beta_n)$  is a sequence not contained in the unit interval Weyl's Theorem becomes:  $(\beta_n)$  is uniformly distributed modulo 1 iff*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h \beta_n} = 0 \text{ for } h = 1, 2, \dots \quad (2.2)$$

#### Proof of Theorem 2.1.1

Throughout this proof we will let  $(\alpha_n)$  be a sequence satisfying the hypothesis of Theorem 2.1.1. To prove the sufficiency of (2.1) we assume that (2.1) holds and let  $f(x) = \chi_{[a,b]}$  be the characteristic function of  $[a, b]$ . That is

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \leq x < b \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\phi_N(a, b)}{N} &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \chi_{[a,b]}(\alpha_n) \\ &= \int_0^1 \chi_{[a,b]}(x) dx \\ &= b - a \end{aligned} \quad (2.3) \quad (\text{using 2.1})$$

Thus  $(\alpha_n)$  is uniformly distributed in the unit interval.

To prove the necessity of (2.1) we assume that  $(\alpha_n)$  is uniformly distributed in the unit interval. (1.1) implies that (2.1) holds for any characteristic function  $f$ . Since (2.1) is a linear equation, it will also hold for any step function  $f$  in  $[0, 1)$ .

For a Riemann integrable function  $f$  we can always find two step functions,  $f_1$  and  $f_2$  with  $f_1 \leq f \leq f_2$  and  $\int_0^1 (f_2(x) - f_1(x)) dx < \epsilon$ . Since  $f_1$  satisfies (2.1), we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_1(\alpha_n) = \int_0^1 f_1(x) dx > \int_0^1 f(x) dx - \epsilon$$

Hence for large enough  $N$ ,

$$\frac{1}{N} \sum_{n=1}^N f_1(\alpha_n) > \int_0^1 f(x) dx - 2\epsilon$$

and since  $f \geq f_1$  then:

$$\frac{1}{N} \sum_{n=1}^N f(\alpha_n) > \int_0^1 f(x) dx - 2\epsilon \quad (2.4)$$

Similarly, using  $f_2$  and taking  $N$  large enough we get:

$$\frac{1}{N} \sum_{n=1}^N f(\alpha_n) < \int_0^1 f(x) dx + 2\epsilon \quad (2.5)$$

Combining (2.4) and (2.5) shows that for  $N$  large enough every Riemann integrable function  $f$  satisfies

$$\left| \frac{1}{N} \sum_{n=1}^N f(\alpha_n) - \int_0^1 f(x) dx \right| < 2\epsilon \quad (2.6)$$

which shows the necessity of 2.1. ■

### **Proof of Theorem 2.1.2**

For a sequence  $(\beta_n)$  not necessarily in the unit interval let  $\alpha_n$  denote the fractional part of  $\beta_n$ . The necessity of (2.2) will follow directly from (2.1). Assume that  $(\beta_n)$  is uniformly distributed modulo 1 and thus  $(\alpha_n)$  is uniformly distributed in the unit interval and so obeys (2.1). Hence, letting  $f(x) = e^{2\pi i h x}$  in (2.1) where  $h$  is an integer,  $h \neq 0$  we have:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h \beta_n} &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h \alpha_n} \\ &= \int_0^1 e^{2\pi i h x} dx \\ &= 0 \end{aligned}$$

Thus (2.2) is a necessary condition for a sequence  $(\beta_n)$  to be u.d. mod 1.

To prove the sufficiency of (2.2) will show that if (2.2) holds for every integer  $h \neq 0$  then (2.1) is satisfied and so the fractional parts of the  $\beta_n$ 's are uniformly distributed in the unit interval. This means that the  $\beta_n$ 's are uniformly distributed modulo 1.

Firstly we observe that  $f(x) = 1$  satisfies (2.1) since

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N 1 = 1 = \int_0^1 dx$$

Assuming (2.2) we get

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h \beta_n} &= \lim_{N \rightarrow \infty} \frac{1}{N} \left( \sum_{n=1}^N \cos 2\pi h \alpha_n + i \sum_{n=1}^N \sin 2\pi h \alpha_n \right) \\ &= 0 \end{aligned} \quad (2.7)$$

Equating real and imaginary parts in (2.7) we see that  $f_{1h}(x) = \cos 2\pi h x$  and  $f_{2h}(x) = \sin 2\pi h x$ , both satisfy (2.1). Once again using the fact that (2.1) is a linear equation we can deduce that (2.1) holds for any trigonometric polynomial of the form

$$a_0 + (a_1 \cos 2\pi x + b_1 \sin 2\pi x) + \cdots + ((a_m \cos 2\pi m x + b_m \sin 2\pi m x) + \cdots$$

The Weierstrass approximation theorem tells us that any continuous function  $f$  of period 1 can be approximated by a trigonometric polynomial of the form above. So, given  $\epsilon > 0$  there exists a trigonometric polynomial  $f_\epsilon$  with  $|f - f_\epsilon| < \epsilon$ . We set  $f_1 = f_\epsilon - \epsilon$  and  $f_2 = f_\epsilon + \epsilon$ . Thus  $f_1 \leq f \leq f_2$  and  $\int_0^1 (f_2(x) - f_1(x)) dx < 2\epsilon$ . Through a similar line of argument to that used to show the necessity of (2.1) we can show that  $f$  satisfies (2.1) and hence (2.2) is a sufficient condition for a sequence  $(\beta_n)$  to be uniformly distributed modulo 1. ■

## 2.1.2 Féjer's Theorem

**Theorem 2.1.3** *Let  $f(x)$  be a function defined for  $x > 1$  that is differentiable for  $x > x_0$ . If  $f'(x) \rightarrow 0$  monotonically as  $x \rightarrow \infty$  and if  $\lim_{x \rightarrow \infty} x|f'(x)| = \infty$  then  $(f(n))$  is uniformly distributed modulo 1.*

Theorem 2.1.3 actually follows as a corollary from the following theorem:

**Theorem 2.1.4** *If  $f(n)$  is a sequence of real numbers such that  $\Delta f(n) = f(n+1) - f(n)$  is monotone as  $n$  increases and in addition*

$$\lim_{n \rightarrow \infty} \Delta f(n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} n|\Delta f(n)| = \infty$$

*then  $(f(n))$  is uniformly distributed modulo 1.*

For a proof of this theorem see [6] page 13.

### Proof of Theorem 2.1.3

The mean value theorem implies that if  $f$  satisfies the criteria of Theorem 2.1.3 then  $\Delta f(n)$  satisfies the conditions of Theorem 2.1.4 for sufficiently large  $n$ . The finitely many exceptional terms do not influence the uniform distribution modulo 1 of the sequence. ■

### 2.1.3 Van der Corput's Difference Theorem

Van der Corput's Difference Theorem discusses the distribution of a sequence based on the distribution of “differences” between terms of the sequence. It essentially allows us to replace a sequence by its derivative and thus will be a very useful tool in discussing the distribution of a sequences which have one of their higher order derivatives being uniformly distributed.

**Theorem 2.1.5** *Let  $(x_n)$  be a sequence of real numbers. If for every positive integer  $h$  the sequence  $(x_{n+h} - x_n)$  is u.d. mod 1 then  $(x_n)$  is uniformly distributed modulo 1.*

To prove the difference theorem we first need the following inequality.

**Lemma 2.1.6** *Let  $u_1 \cdots u_N$  be complex numbers and let  $H$  be an integer with  $1 \leq H \leq N$ . Then*

$$H^2 \left| \sum_{n=1}^N u_n \right|^2 \leq H(N+H-1) \sum_{n=1}^N |u_n|^2 + 2(N+H-1) \sum_{h=1}^{H-1} (H-h) \Re \sum_{n=1}^{N-h} u_n \bar{u}_{n+h}$$

### Proof of Lemma 2.1.6

Define  $u_n = 0$  for all  $n < 0$  and all  $n > N$ . Then we have

$$H \sum_{n=1}^N u_n = \sum_{p=1}^{N+H-1} \sum_{h=1}^{H-1} u_{p-h}$$

The Cauchy-Schwartz inequality states that

$$\sum_{n=1}^n a_n b_n \leq \left( \sum_{p=1}^P |a_p|^2 \right)^{\frac{1}{2}} \left( \sum_{p=1}^P |b_p|^2 \right)^{\frac{1}{2}} \quad (2.8)$$

Letting  $a_n = 1$  and  $b_n = \sum_{h=0}^{H-1} u_{p-h}$  in (2.8) gives:

$$\begin{aligned} H^2 \left| \sum_{n=1}^N u_n \right|^2 &= \left| \sum_{p=1}^{N+H-1} \underbrace{1}_{a_n} \underbrace{\sum_{h=1}^{H-1} u_{p-h}}_{b_n} \right|^2 \\ &\leq (N+H-1) \sum_{p=1}^{N+H-1} \left| \sum_{h=0}^{H-1} u_{p-h} \right|^2 \quad \text{by (2.8)} \\ &= (N+H-1) \sum_{p=1}^{N+H-1} \left( \sum_{r=0}^{H-1} u_{p-r} \right) \left( \sum_{s=0}^{H-1} \bar{u}_{p-s} \right) \\ &= (N+H-1) \sum_{p=1}^{N+H-1} \sum_{h=0}^{H-1} |u_{p-h}|^2 \\ &\quad + 2(N+H-1) \Re \sum_{p=1}^{N+H-1} \sum_{r,s=0 \leq s < r}^{H-1} u_{p-r} \bar{u}_{p-s} \\ &= (N+H-1) (\Sigma_1 + 2\Re \Sigma_2) \end{aligned}$$

Now  $\Sigma_1 = H \sum_{n=1}^N |u_n|^2$ .  $\Sigma_2$  contains terms of the form  $u_n \bar{u}_{n+h}$  where  $n = 1 \cdots N$  and  $h = r-s = 1 \cdots H-1$ . For fixed  $n$  and  $h$  the possible choices for the pair  $(r, s)$  are given explicitly by  $(h, 0), (h-1, 1), \dots, (H-1, H-h-1)$  and for each such choice the value of  $p$  is unique. Hence there are exactly  $H-h$  occurrences of  $u_n \bar{u}_{n+h}$  in  $\Sigma_2$ . Thus we have

$$\Sigma_2 = \sum_{h=1}^{H-1} (H-h) \sum_{n=1}^N u_n \bar{u}_{n+h}$$

And since  $u_n = 0$  for  $n > N$  the summation over  $n$  can be restricted to summation over  $1 \leq n \leq N-h$ , and Lemma 2.1.6 follows.  $\blacksquare$

### Proof of Theorem 2.1.5

Let  $m$  be a fixed non zero integer. We apply Lemma 2.1.6 with  $u_n = e^{2\pi i m x_n}$

and dividing by  $H^2 N^2$  we get:

$$\left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i m x_n} \right|^2 \leq \frac{N+H-1}{HN} + 2 \sum_{h=1}^{H-1} \frac{(N+H-1)(H-h)(N-h)}{H^2 N^2} \left| \frac{1}{N-h} \sum_{n=1}^{N-h} e^{2\pi i m (x_n - x_{n+h})} \right| \quad (2.9)$$

Now since the sequence  $(x_{n+h} - x_n)$  is u.d. mod 1 for every  $h \geq 1$ , for such  $h$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{N-h} \sum_{n=1}^{N-h} e^{2\pi i m (x_n - x_{n+h})} = 0 \quad (2.10)$$

By (2.9) and (2.10) we get

$$\limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i m x_n} \right|^2 \leq \frac{1}{H^2} \quad (2.11)$$

and since (2.11) holds for every positive integer  $H$  we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i m x_n} = 0$$

and thus  $(x_n)$  is u.d. mod 1. ■

#### 2.1.4 Generalised Van Der Corput's Difference Theorem

Although Van der Corput's Difference theorem is a very powerful tool, it is not particularly useful for sequences whose differences are not u.d. mod 1. There are sequences whose derivative is "almost" u.d. mod 1, about which we would like to say something. We say that a sequence  $(x_n)$  is "almost" u.d. if  $(x_n)$  does not satisfy Weyl's criterion, but has the property that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h x_n} \right| \rightarrow 0$$

as  $h \rightarrow \infty$ . We develop the generalised Van Der Corput's Difference theorem stated below to deal with sequences whose differences are "almost" uniformly distributed modulo 1.

**Theorem 2.1.7** *Let  $(x_n)$  be a sequence of real numbers. If for every positive integer  $h$  we have*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i m(x_{n+h} - x_n)} \right| \leq \frac{K}{h^\alpha} \quad (2.12)$$

*for some  $\alpha > 0$  then  $(x_n)$  is uniformly distributed modulo 1.*

If  $\alpha \geq 1$  replace  $\alpha$  by  $\alpha'$  with  $\alpha' < 1$ . Certainly (2.12) still holds if  $\alpha$  is replaced by  $\alpha'$  and so we can restrict our proof to deal with cases when  $\alpha < 1$ . In order to prove the theorem we will first need the following proposition:

**Proposition 2.1.8** *If  $N > H$  and  $\alpha < 1$  then*

$$\frac{1}{N} \sum_{h=1}^{H-1} \frac{(N+H-1)(H-h)(N-h)}{H^2 N^2 h^\alpha} \leq K H^{-\alpha}$$

**Proof**

Firstly we observe that for  $\alpha < 1$

$$\begin{aligned} \sum_{h=1}^{H-1} h^{-\alpha} &= 1 + \underbrace{\frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{(2^{\frac{1}{\alpha}})^\alpha}}_{\leq 2^{\frac{1}{\alpha} \cdot 1}} + \underbrace{\frac{1}{(2^{\frac{1}{\alpha}} + 1)^\alpha} + \cdots + \frac{1}{(2^{\frac{2}{\alpha}})^\alpha}}_{\leq 2^{\frac{2}{\alpha} \cdot \frac{1}{2}}} + \cdots \\ &\leq 2^{\frac{1}{\alpha} \cdot 1} + (2^{\frac{1}{\alpha}})^2 \cdot \frac{1}{2} + (2^{\frac{1}{\alpha}})^3 \cdot \frac{1}{4} + \cdots + (2^{\frac{1}{\alpha}})^{\lceil \log_2 H^\alpha \rceil} \cdot \frac{1}{2^{\lceil \log_2 H^\alpha \rceil - 1}} \\ &= 2^{\frac{1}{\alpha}} \left( 1 + 2^{(\frac{1}{\alpha} - 1)} + 2^{2(\frac{1}{\alpha} - 1)} + \cdots + 2^{(\log_2 H^\alpha - 1)(\frac{1}{\alpha} - 1)} \right) \\ &= 2^{\frac{1}{\alpha}} \sum_{j=0}^{\log_2 H^{\frac{1}{\alpha}} - 1} 2^{j(\frac{1}{\alpha} - 1)} \\ &= 2^{\frac{1}{\alpha}} \frac{2^{(\frac{1}{\alpha} - 1) \log_2 H^\alpha} - 1}{2^{(\frac{1}{\alpha} - 1)} - 1} \\ &= \frac{2^{\frac{1}{\alpha}}}{2^{(\frac{1}{\alpha} - 1)} - 1} \left( 2^{\log_2 H^{1-\alpha}} - 1 \right) \\ &= K (H^{1-\alpha} - 1) \\ &\leq K H^{1-\alpha} \end{aligned} \quad (2.13)$$

Then

$$\begin{aligned}
\sum_{h=1}^{H-1} \frac{(N+H-1)(H-h)(N-h)}{H^2 N^2 h^\alpha} &< \sum_{h=1}^{H-1} \frac{2NHN}{H^2 N^2 h^\alpha} \\
&= \frac{2}{H} \sum_{h=1}^{H-1} \frac{1}{h^\alpha} \\
&\leq \frac{KH^{1-\alpha}}{H} \quad (\text{Using (2.13)}) \\
&= KH^{-\alpha}
\end{aligned} \tag{2.14}$$

And so the proposition is proved.  $\blacksquare$

**Proof of Theorem 2.1.7**

We now re-work the proof of Van Der Corput's Difference Theorem. Taking lim sups as  $N \rightarrow \infty$  equation (2.9) becomes:

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i m x_n} \right|^2 &\leq \limsup_{N \rightarrow \infty} \left( \frac{N+H-1}{HN} + \right. \\
&\quad \left. 2 \sum_{h=1}^{H-1} \frac{(N+H-1)(H-h)(N-h)}{H^2 N^2} \left| \frac{1}{N-h} \sum_{n=1}^{N-h} e^{2\pi i m (x_n - x_{n+h})} \right| \right) \\
&\leq \limsup_{N \rightarrow \infty} \left( \frac{N+H-1}{HN} + 2 \sum_{h=1}^{H-1} \frac{(N+H-1)(H-h)(N-h)}{H^2 N^2 h^\alpha} \right) \\
&\quad (\text{Using the hypothesis}) \\
&\leq \limsup_{N \rightarrow \infty} \left( \frac{N+H-1}{HN} + \frac{K}{H^{-\alpha}} \right) \\
&\quad (\text{Using Proposition 2.1.8}) \\
&\leq \frac{K}{H^{-\alpha}}
\end{aligned} \tag{2.15}$$

Thus for every positive integer  $H$  we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i m x_n} \right| \leq \frac{K}{H^{\alpha/2}} \tag{2.16}$$

which means that  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i m x_n} = 0$  and so  $(x_n)$  is u.d. mod 1.  $\blacksquare$



## 2.2 Tools

In this section we outline some of the tools which will be used often in the course of this report. Generally the tools simplify the approximation of exponential sums. They include estimating sums by integrals and then simplifying these integrals (Sections 2.2.1, 2.2.2 and 2.2.3), approximating exponential sums of sequences by exponential sums of nearby simpler sequences (Section 2.2.4) and ignoring the first few terms of a sequence (Section 2.2.5).

### 2.2.1 The Euler Summation Formula

Euler's summation formula provides a method for us to change sums into integrals. This is particularly useful as it is far easier for us to evaluate the integrals than it is to evaluate the sums. The formula states that:

$$\sum_{n=M}^N F(n) = \int_M^N F(x)dx + \frac{1}{2}(F(M) + F(N)) + \int_M^N \left( \{x\} - \frac{1}{2} \right) F'(x)dx \quad (2.17)$$

For a proof of this see e.g. [6] page 25.

Given a function  $f(n)$  we let  $F(x) = e^{2\pi i h f(x)}$ . After dividing both sides of (2.17) by  $N$  and taking the limit as  $N \rightarrow \infty$  we arrive at the following formula:

$$\begin{aligned} \limsup_{N \rightarrow \infty} \left| \sum_{n=M}^N e^{2\pi i h f(n)} \right| &= \left| \limsup_{N \rightarrow \infty} \underbrace{\frac{1}{N} \int_M^N e^{2\pi i h f(x)} dx}_{\text{First Term}} + \limsup_{N \rightarrow \infty} \frac{1}{2N} (e^{2\pi i h f(M)} + e^{2\pi i h f(N)}) \right. \\ &\quad \left. + \limsup_{N \rightarrow \infty} \underbrace{\frac{1}{N} \int_M^N \left( \{x\} - \frac{1}{2} \right) (2\pi i h) f'(x) e^{2\pi i h f(x)} dx}_{\text{Third Term}} \right| \\ &\leq \limsup_{N \rightarrow \infty} \frac{|\text{First Term}|}{N} + \limsup_{N \rightarrow \infty} \frac{1}{2N} \underbrace{|e^{2\pi i h f(M)} + e^{2\pi i h f(N)}|}_{<2} \\ &\quad + \limsup_{N \rightarrow \infty} \frac{|\text{Third Term}|}{N} \\ &= \limsup_{N \rightarrow \infty} \frac{|\text{First Term}|}{N} + \limsup_{N \rightarrow \infty} \frac{|\text{Third Term}|}{N} \end{aligned} \quad (2.18)$$

Note also that:

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|\text{Third Term}|}{N} &= \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \int_M^N \left( \{x\} - \frac{1}{2} \right) (2\pi i h) f'(x) e^{2\pi i h f(x)} dx \right| \\
&\leq \limsup_{N \rightarrow \infty} \frac{K}{N} \int_M^N \left| \{x\} - \frac{1}{2} \right| |f'(x)| |e^{2\pi i h f(x)}| dx \\
&\leq \limsup_{N \rightarrow \infty} \frac{K}{N} \int_M^N |f'(x)| dx
\end{aligned} \tag{2.19}$$

If we can show that  $\limsup_{N \rightarrow \infty} \frac{|\text{First Term}|}{N} = 0$  and  $\limsup_{N \rightarrow \infty} \frac{|\text{Third Term}|}{N} = 0$  this implies that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=M}^N e^{2\pi i h f(x)} = 0$$

### 2.2.2 Integration By Parts

We will frequently want to estimate the size of an integral of a function of the form  $e^{2\pi i h f(x)}$ . To do this we will often use the trick of integrating by parts. Let  $A$  be the interval (or set of intervals) over which we are integrating. Then we get

$$\begin{aligned}
\left| \int_A e^{2\pi i h f(x)} dx \right| &= \left| \int_A \frac{d}{dx} (e^{2\pi i h f(x)}) \frac{dx}{2\pi i h f'(x)} \right| \\
&\leq \frac{1}{h} \left| \left[ \frac{e^{2\pi i h f(x)}}{f'(x)} \right]_A \right| + \frac{1}{h} \left| \int_A \frac{e^{2\pi i h f(x)} f''(x)}{(f'(x))^2} dx \right| \\
&\leq \frac{1}{h} \left[ \frac{1}{|f'(x)|} \right]_A + \frac{1}{h} \int_A \left| \frac{f''(x)}{(f'(x))^2} \right| dx
\end{aligned} \tag{2.20}$$

It will often be easier for us to approximate this new expression.

### 2.2.3 Integrating Functions Whose Derivative can be Simplified

**Proposition 2.2.1** *If  $f(x)$  is a function whose derivative can be decomposed into  $f'(x) = A(x) + B(x)$  with  $\epsilon|A(x)| \geq |B(x)|$  and  $S$  is an interval or (set of intervals) over which we are integrating then*

$$\frac{1}{h} \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \frac{dx}{f'(x)} \right| \leq \frac{1}{h} \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \frac{dx}{A(x)} \right| + \epsilon K |S| \tag{2.21}$$

**Proof**

$$\begin{aligned}
\frac{1}{h} \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \frac{dx}{f'(x)} \right| &= \frac{1}{h} \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \frac{dx}{A(x) + B(x)} \right| \\
&= \frac{1}{h} \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \left( \frac{1}{A(x)} - \frac{B(x)}{(A(x) + B(x))A(x)} \right) dx \right| \\
\frac{1}{h} &\leq \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \frac{dx}{A(x)} \right| \\
&\quad + \frac{1}{h} \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \underbrace{\frac{1}{A(x) + B(x)}}_{f'(x)} \underbrace{\frac{B(x)}{A(x)}}_{\leq \epsilon} dx \right| \\
&\leq \frac{1}{h} \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \frac{dx}{A(x)} \right| \\
&\quad + \frac{1}{h} \int_S \left| 2\pi i h f'(x) e^{2\pi i h f(x)} \frac{\epsilon}{f'(x)} \right| dx \\
&\leq \frac{1}{h} \left| \int_S \frac{d}{dx} (e^{2\pi i h f(x)}) \frac{dx}{A(x)} \right| + \epsilon K |S|
\end{aligned} \tag{2.22}$$

And so the proposition follows. ■

## 2.2.4 Approximating Sequences by Simpler Sequences

**Theorem 2.2.2** *Let  $(v_n)$  and  $(u_n)$  be sequences. If for every  $0 < \epsilon < 1$  there is an  $N_\epsilon$  such that  $v_n$  can be written as  $v_n = u_n + \epsilon_n$  with  $\epsilon_n < \epsilon \forall n > N_\epsilon$  then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left( \sum_{n=1}^N e^{2\pi i h v_n} - \sum_{n=1}^N e^{2\pi i h u_n} \right) = 0$$

**Proof**

Firstly we observe that using Taylor Expansion gives:

$$\begin{aligned}
|e^{2\pi i h(x+\epsilon)} - e^{2\pi i h x}| &= |\epsilon 2\pi i h e^{2\pi i h x} - \epsilon^2 2\pi^2 h^2 e^{2\pi i h x} + \dots| \\
&\leq K_1 \epsilon + K_2 \epsilon^2 + \dots \\
&\leq K \epsilon \\
&= O(\epsilon)
\end{aligned} \tag{2.23}$$

And thus

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h v_n} - \sum_{n=1}^N e^{2\pi i h u_n} \right| \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N (e^{2\pi i h v_n} - e^{2\pi i h u_n}) \right| \\
&\leq \underbrace{\lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^{N_\epsilon} (e^{2\pi i h v_n} - e^{2\pi i h u_n}) \right|}_{=0} + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=N_\epsilon}^N \underbrace{|e^{2\pi i h v_n} - e^{2\pi i h u_n}|}_{O(\epsilon_n)} \\
&\leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N O(\epsilon_n) \\
&\leq O(\epsilon)
\end{aligned} \tag{2.24}$$

Since (2.24) holds  $\forall \epsilon$  the theorem follows.  $\blacksquare$

The following two corollaries follow as a consequence of the above theorem.

**Corollary 2.2.3** *Let  $(u_n)$  and  $(v_n)$  be sequences which satisfy the hypothesis of Theorem 2.2.2. Then*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h v_n} \right| = \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h u_n} \right| + O(\epsilon)$$

**Proof**

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h v_n} \right| &= \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h v_n} - \sum_{n=1}^N e^{2\pi i h u_n} + \sum_{n=1}^N e^{2\pi i h u_n} \right| \\
&= \underbrace{\lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h v_n} - \sum_{n=1}^N e^{2\pi i h u_n} \right|}_{=0} + \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h u_n} \right| \\
&= \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h u_n} \right|
\end{aligned} \tag{2.25}$$

$\blacksquare$

**Corollary 2.2.4** *Let  $(u_n)$  and  $(v_n)$  be sequences which satisfy the hypothesis of Theorem 2.2.2. Then*

- (i) *If  $(u_n)$  is uniformly distributed modulo 1 then  $(v_n)$  is also uniformly distributed modulo 1.*
- (ii) *If  $(u_n)$  is not uniformly distributed modulo 1 then neither is  $(v_n)$ .*

**Proof**

(i) If  $(u_n)$  is u.d. mod 1 then the lim sup on the right hand side of (2.25) can be replaced by a limit and we get:

$$0 = \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h u_n} \right| = \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h v_n} \right|$$

and thus  $(v_n)$  is also uniformly distributed modulo 1.

(ii) Let  $w_n = u_n - \epsilon_n$ . By the contrapositive of (i) if  $(w_n + \epsilon_n) (= (u_n))$  is not u.d. mod 1 then neither is  $(w_n)$ . Hence if  $(u_n)$  is not u.d. mod 1, then neither is  $(u_n - \epsilon_n)$  and similarly, neither is  $(u_n + \epsilon_n) = (v_n)$ . ■

## 2.2.5 Changing the summation limits

**Proposition 2.2.5**

- (i) *If for every  $\epsilon > 0$  and for every  $h \neq 0$ ,  $\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{\epsilon N}^N e^{2\pi i h f(n)} \right| = O(\epsilon)$  then  $(f(n))$  is uniformly distributed modulo 1.*
- (ii) *Conversely, if for every  $\epsilon > 0$  there is an  $h \neq 0$  such that  $\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{\epsilon N}^N e^{2\pi i h f(n)} \neq 0$  then  $(f(n))$  is not uniformly distributed modulo 1.*
- (iii)  *$(f(n))$  is uniformly distributed modulo 1 iff  $\lim_{N \rightarrow \infty} \frac{\phi_{(\epsilon N, N)}^{f(n)}(a, b)}{N} = (b - a)(1 - \epsilon)$ .*

**Proof**

(i) Pick  $\epsilon > 0$  then using the hypothesis of part (i) we have

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h f(n)} \right| &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^{\epsilon N} e^{2\pi i h f(n)} \right| + \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=\epsilon N}^N e^{2\pi i h f(n)} \right| \\ &= O(\epsilon) \end{aligned}$$

Since we can pick  $\epsilon$  to be arbitrarily small we conclude that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h f(n)} = 0$$

and so  $(f(n))$  is u.d. mod 1.

(ii) Firstly we will show that if  $(f(n))$  is u.d. mod 1 then  $\forall \epsilon > 0$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\epsilon N}^N e^{2\pi i h f(n)} = 0.$$

$$\begin{aligned} \lim_{N \rightarrow \infty} \left| \sum_{n=\epsilon N}^N e^{2\pi i h f(n)} \right| &= \left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h f(n)} - \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{\epsilon N} e^{2\pi i h f(n)} \right| \\ &\leq \underbrace{\lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h f(n)} \right|}_{=0} - \underbrace{\lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^{\epsilon N} e^{2\pi i h f(n)} \right|}_{=0} \\ &= 0 \end{aligned} \tag{2.26}$$

It should be noted that since all the limits on the right hand side of (2.26) exist then the limit on the left hand side must also exist.

The converse of this statement proves (ii).

The statements of (i) and (ii) combine to give an interesting result. They show that if  $\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=\epsilon N}^N e^{2\pi i h f(n)} \right| = O(\epsilon)$  then it must necessarily be zero.

(iii) Firstly if  $(f(n))$  is u.d. mod 1 then  $\forall \epsilon > 0$

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\phi_{(\epsilon N, N)}^{f(n)}(a, b)}{N} &= \lim_{N \rightarrow \infty} \frac{\phi_N^{f(n)}(a, b)}{N} - \lim_{N \rightarrow \infty} \frac{\epsilon \phi_{\epsilon N}^{f(n)}(a, b)}{\epsilon N} \\ &= (b - a)(1 - \epsilon) \end{aligned} \tag{2.27}$$

The converse of (2.27) proves the other direction. ■

## Chapter 3

# Non-Trigonometric Sequences

The first half of this chapter deals with the distribution of specific families of non-trigonometric sequences such as  $n\theta$ ,  $\log n$ ,  $n^\alpha$  and combinations of these. The second half of the chapter deals with general families of sequences for which we know something about their derivatives.

### 3.1 The distribution of $n\theta$

**Proposition 3.1.1** *If  $\theta$  is a rational number then  $(n\theta)$  is not uniformly distributed modulo 1.*

**Proof**

Let  $\theta = \frac{p}{q}$  and let  $h = q$ . Then

$$\begin{aligned} e^{2\pi i h n \theta} &= e^{2\pi i n p} \\ &= e^{2\pi i p} \end{aligned}$$

for all  $n$ . Hence,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h n \theta} \right| &= \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i p} \right| \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N 1 \right| \\ &= 1 \\ &\neq 0 \end{aligned}$$

And thus Weyl's criterion fails. ■

**Proposition 3.1.2** *If  $\theta$  is irrational then  $n\theta$  is uniformly distributed modulo 1.*

**Proof**

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h n \theta} \right| &= \lim_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N (e^{2\pi i h \theta})^n \right| \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \left| \frac{e^{2\pi i h \theta} (1 - e^{2\pi i h N \theta})}{1 - e^{2\pi i h \theta}} \right| \\ &\leq \lim_{N \rightarrow \infty} \frac{1}{N} \left( \frac{1}{|1 - e^{2\pi i h \theta}|} + \frac{|e^{2\pi i h N \theta}|}{|1 - e^{2\pi i h \theta}|} \right) \\ &= \lim_{N \rightarrow \infty} \frac{K}{N} \\ &= 0 \end{aligned}$$

Note: The above proof only works as  $|1 - e^{2\pi i h \theta}| \neq 0$  for any value of  $h$ , since  $\theta$  is irrational. ■

It is not too hard to prove the following Proposition as a consequence of Propositions 3.1.1 and 3.1.2 and Van Der Corput's difference theorem.



**Proposition 3.1.3** *If  $P(n) = a_0 + a_1n + \cdots + a_kn^k$ ,  $a_i \in \mathbb{R}$   $i = 0, \dots, k$  then  $P(n)$  is uniformly distributed modulo 1 iff at least one of the  $a_i$ 's  $i = 1, \dots, k$  is irrational.*

**Proof**

For a proof of this see [6] Theorem 3.2 Page 27. ■

## 3.2 The distribution of $\log n$

**Proposition 3.2.1**  *$(\log n)$  is not uniformly distributed modulo 1.*

**Proof**

We will show that (2.2) does not hold for the case when  $h = 1$ . To do this we will use Euler's summation formula.

We look at the terms on the right hand side of equation 2.17.

$$\begin{aligned} \frac{\text{First term}}{N} &= \frac{1}{N} \int_1^N e^{2\pi i \log x} dx \\ &= \frac{1}{N} [xe^{2\pi i \log x}]_1^N \\ &= \frac{K}{N} (Ne^{2\pi i \log N} - 1) \\ &= Ke^{2\pi i \log N} \end{aligned} \tag{3.1}$$

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{|\text{Third term}|}{N} &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \int_1^N \left| \frac{2\pi i}{x} \right| dx \\ &= \limsup_{N \rightarrow \infty} \frac{K}{N} [\log x]_1^N \\ &= \limsup_{N \rightarrow \infty} \frac{K \log N}{N} \\ &= 0 \end{aligned} \tag{3.2}$$

Thus the second and third terms tend to zero as  $N \rightarrow \infty$ , and (3.1) shows that the first term doesn't converge as  $N \rightarrow \infty$ . This means that  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i \log n}$  does not exist and so cannot be equal to zero and thus  $\beta_n = \log(n)$  doesn't satisfy (2.2). By the necessity of (2.2) we conclude that  $(\log n)$  is not uniformly distributed modulo 1. ■

### 3.3 The distribution of $\sum_j c_j n^{\alpha_j}$

In this section we will discuss the distribution of  $(\sum_j c_j n^{\alpha_j})$ . As a result of this discussion we will be able to conclude that  $(n^\alpha)$  is uniformly distributed modulo 1 for any positive non-integer valued  $\alpha$ . (Corollary 3.3.2). It should be noted that the method used to integrate various functions in this section is not necessarily the easiest or most efficient method, however it is the same method as will be necessary in chapter 4. We have used this method here to introduce the reader to arguments which will be used repeatedly later in the report.

For  $m \in \mathbb{Z}$  define  $f_m(x) = \sum_j c_j x^{\alpha_j}$  where the sum consists of finitely many terms,  $c_j \in \mathbb{R}$  and  $\alpha_j < m$ . Let  $k$  be the subscript of the largest  $\alpha_j$  in  $f_m(x)$ . Ensure that  $\alpha_k > 0$ ,  $\alpha_k \notin \mathbb{Z}$  and  $c_k \neq 0$ .

**Proposition 3.3.1** *For  $f_m(x)$  defined as above,  $(f_m(n))$  is uniformly distributed modulo 1.*

#### Proof

We will prove this proposition by induction on  $m$ . We first consider the case where  $m = 1$ . Let  $k$  be chosen as above and let  $k'$  be the subscript of the second largest  $\alpha_j$ . Define  $M = \max_j \{|c_j \alpha_j|\}$  and let  $T$  be the number of terms in the sum of  $f_1(x)$ . Pick  $N > \frac{1}{\epsilon} \left( \frac{MT}{\epsilon |c_k \alpha_k|} \right)^{\frac{1}{\alpha_k - \alpha_{k'}}$ . This will mean that for  $x > \epsilon N$  we have  $\epsilon c_k \alpha_k x^{\alpha_k - 1} < MT x^{\alpha_{k'} - 1}$  which will be necessary during the proof. Then

$$\begin{aligned} f_1'(x) &= \sum_j c_j \alpha_j x^{\alpha_j - 1} \\ &= \underbrace{c_k \alpha_k x^{\alpha_k - 1}}_{A(x)} + \underbrace{\sum_{j, j \neq k} c_j \alpha_j x^{\alpha_j - 1}}_{B(x)} \end{aligned}$$

And for  $x > \epsilon N$  we have

$$\begin{aligned} |B(x)| &\leq \sum_{j, j \neq k} |c_j \alpha_j x^{\alpha_j - 1}| \\ &\leq \sum_{j, j \neq k} |c_j \alpha_j| x^{\alpha_{k'} - 1} \\ &\leq MT x^{\alpha_{k'} - 1} \\ &\leq \epsilon |c_k \alpha_k| x^{\alpha_k - 1} \\ &\quad \text{(Due to our choice of } N\text{)} \\ &= \epsilon |A(x)| \end{aligned} \tag{3.3}$$

We will thus be able to apply Proposition 2.2.1.

In order to show that  $f_1(n)$  is u.d. mod 1 we will consider  $\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=\epsilon N}^N e^{2\pi i h f_1(n)}$  and show that this is of order  $\epsilon$ . Invoking Proposition 2.2.5 will then give the desired result. We now look at the terms of the Euler summation formula given in section 2.2.1.

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|\text{First Term}|}{N} &= \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \int_{\epsilon N}^N e^{2\pi i h \sum_j c_j x^{\alpha_j}} dt \right| \\
&= \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \int_{\epsilon N}^N \frac{d}{dx} \left( e^{2\pi i h \sum_j c_j x^{\alpha_j}} \right) \left( \sum_j \alpha_j c_j x^{\alpha_j-1} \right)^{-1} dx \right| \\
&\leq \limsup_{N \rightarrow \infty} \frac{K}{N} \left| \int_{\epsilon N}^N \frac{d}{dx} \left( e^{2\pi i h \sum_j c_j x^{\alpha_j}} \right) (\alpha_k c_k x^{\alpha_k-1})^{-1} dx \right| + \limsup_{N \rightarrow \infty} \frac{\epsilon K' N}{N} \\
&\quad \text{(Using (3.3) and Proposition 2.2.1)} \\
&\leq \limsup_{N \rightarrow \infty} \frac{K}{N} \left( \left| \left[ \frac{1}{\alpha_k c_k x^{\alpha_k-1}} \right]_{\epsilon N}^N \right| + \int_{\epsilon N}^N |x^{-\alpha_k}| dx \right) + \epsilon K' \\
&\leq \limsup_{N \rightarrow \infty} \left( \frac{K_1}{N} [x^{1-\alpha_k}]_M^N + \frac{K_2}{N} \int_M^N x^{-\alpha_k} dx \right) + \epsilon K' \\
&\leq \limsup_{N \rightarrow \infty} \frac{K}{N} (N^{1-\alpha_k} + N^{1-\alpha_k}) + \epsilon K' \\
&= \limsup_{N \rightarrow \infty} K N^{-\alpha_k} + \epsilon K' \\
&= \epsilon K' \\
&= O(\epsilon)
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|\text{Third term}|}{N} &= \limsup_{N \rightarrow \infty} \frac{K}{N} \int_{\epsilon N}^N \left| \sum_j \alpha_j c_j x^{\alpha_j-1} \right| dx \\
&\leq \limsup_{N \rightarrow \infty} \frac{KT}{N} \int_{\epsilon N}^N x^{(\alpha_k-1)} dx \\
&= \limsup_{N \rightarrow \infty} \frac{K}{N} [x^{\alpha_k}]_{\epsilon N}^N \\
&\leq \limsup_{N \rightarrow \infty} \frac{K N^{\alpha_k}}{N} \\
&= 0
\end{aligned} \tag{3.5}$$

Thus combining equations (3.4) and (3.5) and using the results of section 2.2.1 and Proposition 2.2.5 we can conclude that  $(f_1(n))$  is u.d. mod 1.

Now assume that  $(f_b(n))$  is u.d. mod 1 for all  $b \leq m$ , and consider the distribution of  $(f_m(n))$ . Once again we let  $k$  be the subscript of the largest  $\alpha_j$  in  $f_m(n)$ . We can assume that the  $c_k \neq 0$  as otherwise we could just use the induction hypothesis to prove the u.d. mod 1 of the sequence  $(f_m(n))$ . Using Taylor expansion we get:

$$\begin{aligned}
& f_m(n+h) - f_m(n) \\
&= hf_m^{(1)}(n) + \frac{h^2}{2}f_m^{(2)}(n) + \cdots + \frac{h^{m-1}}{(m-1)!}f_m^{(m-1)}(n) + O(n^{\alpha_k-1}) \\
&\quad \text{(Where the constant in the } O \text{ term depends on the value of } h.) \\
&= h \sum_j \alpha_j c_j n^{\alpha_j-1} + \frac{h^2}{2} \sum_j (\alpha_j - 1) \alpha_j c_j n^{\alpha_j-2} + \cdots \\
&\quad + \frac{h^{m-1}}{(m-1)!} (Kn^{\alpha_k} + l.o.t.) + O(n^{\alpha_k-1}) \\
&= f_{m-1}^*(n) + f_{m-2}^*(n) + \cdots + f_1^*(n) + \underbrace{O(n^{\alpha_k-1})}_{< \epsilon} \\
&\quad \text{(Where we can make the last term as small as we want.)} \\
&= f_{m-1}^{**}(n) + O(\epsilon)
\end{aligned} \tag{3.6}$$

Now the leading co-efficient of  $f_{m-1}^{**}(n)$  is  $\alpha_k c_k$  which is non-zero and the leading power of  $f_{m-1}^{**}(n)$  is  $(\alpha_k - 1) \notin \mathbb{Z}$  so  $f_{m-1}^{**}(n)$  satisfies the induction hypothesis and so is u.d. mod 1. Hence by (2.1.5)  $(f_m(n))$  is u.d. mod 1 for all  $m$ . ■

**Corollary 3.3.2** *If  $0 < \alpha$ ,  $\alpha \notin \mathbb{Z}$  then  $(n^\alpha)$  is uniformly distributed modulo 1.*

**Proof**

Simply let  $m = \lceil \alpha \rceil$  and the corollary follows. ■

### 3.4 The distribution of $(\sum c_j n^{\alpha_j} \log^{\tau_j} n)$

For  $k > 0$  let  $f_k(n) = \sum c_j n^{\alpha_j} \log^{\tau_j} n$  where  $0 < \alpha_j < k$ , each  $\alpha_j$  is unique and at least one  $\alpha_j$  is positive. In addition we require that  $c_j \neq 0$ .

**Proposition 3.4.1** *For  $f_k(n)$  defined as above,  $(f_k(n))$  is uniformly distributed modulo 1 for every  $k$ .*

**Proof**

We will prove Proposition 3.4.1 by induction on  $k$  using a similar argument to that invoked in the previous section. To deal with the distribution of  $f_1(n)$  we will invoke (Féjer's Theorem)

$$\begin{aligned} f'_1(x) &= \sum \alpha_j c_j x^{\alpha_j-1} \log^{\tau_j} x + \tau_j c_j x^{\alpha_j-1} \log^{\tau_j-1} x \\ &\rightarrow 0 \text{ monotonically as } x \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} x|f'_1(x)| &= \left| \sum \alpha_j c_j x^{\alpha_j} \log^{\tau_j} x + \tau_j c_j x^{\alpha_j} \log^{\tau_j-1} x \right| \\ &\rightarrow \infty \text{ as } x \rightarrow 0 \end{aligned}$$

So  $f_1(n)$  satisfies the hypothesis of Theorem 2.1.3 (Féjer's Theorem) and Proposition 3.4.1 holds for  $k = 1$ .

Assume that Proposition 3.4.1 holds for all  $k \leq m$ , and consider the case when  $k = m + 1$ . Once again we let  $f_k(n) = \sum c_j n^{\alpha_j} \log^{\tau_j} n$  where  $\alpha_j < k$ . Let

$$\alpha = \max_j \alpha_j \quad \text{and} \quad \tau = \max_{j, \alpha_j = \alpha} \tau_j$$

Finally, let  $r$  be the index such that  $\alpha_r = \alpha$  and  $\tau_r = \tau$ . Using Taylor expansion gives:

$$\begin{aligned} &f_{m+1}(n+h) - f_{m+1}(n) \\ &= h f_{m+1}^{(1)}(n) + \frac{h^2}{2} f_{m+1}^{(2)}(n) + \cdots + \frac{h^k}{k!} f_{m+1}^{(k)}(n) + O(n^s) \\ &\quad \text{where } s < 0 \text{ and the constant in the } O \text{ term depends on the value of } h \\ &= f_m^*(n) + \underbrace{\sum_{i=0}^{m-1} \sum_j C_{ij} n^{\alpha_j-i} \log^{\tau_{bij}} n}_{< \epsilon} + O(n^s) \end{aligned}$$

where we can make the last term as small as we want

(3.7)

Now the coefficient of  $n^{\alpha_r-1} \log^{\tau_r} n$  is non-zero and so  $f_m^*(n)$  satisfies the induction hypothesis and thus is u.d. mod 1. Hence by (2.1.5)  $(\sum c_j n^{\alpha_j} \log^{\tau_j} n)$  is u.d. mod 1 ■

## 3.5 Distribution of “slow sequences”

In this section we look at sequences which are increasing slowly. The theorem which follows is actually proved more generally and succinctly as a corollary to Theorem 3.6.1. The reason for including this proof is that a very different approach to proving the theorem is used here. Generally when proving the uniform distribution or otherwise of a sequence we try to place either a lower or upper bound on its exponential sum, and the most common way of doing that is by changing the sums into integrals and then integrating. In this section we deal directly with the terms of the exponential sum, and show that the imaginary part of the sum does not go to zero. Thus we obtain a much better understanding of how the exponential sums are behaving.

**Theorem 3.5.1** *If  $0 < f'(x) < \frac{1}{2\pi x}$  and  $f''(x) < 0$  then  $f(n)$  is not uniformly distributed modulo 1.*

### 3.5.1 Outline of Proof

To begin with we observe that if  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h f(n)} = 0$  then both the real and imaginary parts of this sum go to zero. In this proof we will show that  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \sin 2\pi f(n) \neq 0$  and thus Weyl's criterion fails for  $h = 1$  and hence the sequence is not u.d. mod 1.

To do this we will show that with each rotation about the unit circle the sum of the imaginary terms accrued is strictly negative. This will be done by breaking one rotation around the circle into four components, with the  $k$ th component being when  $\theta \in [\frac{k\pi}{2}, \frac{(k+1)\pi}{2}]$ ,  $k = 1, 2, 3, 4$ .

Terms in the 1st and 2nd components have positive imaginary parts, while terms in the 3rd and 4th components have negative imaginary parts. We will show that the negative terms in the 3rd component cancel out with the positive terms in the 1st component and the negative terms in the 4th component cancel out the positive terms in the 2nd one. In addition we will show that there are many extra negative terms left over after this cancellation process, and so the total sum around the circle is negative. We will need to take special care when dealing with terms whose imaginary part is very close to 1 to ensure that they are cancelled out correctly.

#### Proof

Pick  $N_0$  to be large and w.l.o.g. assume  $f(N_0) = k$  for  $k \in \mathbb{Z}$

Let  $N_1$  be the last  $n > N_0$  with  $f(n) - f(N_0) < \frac{1}{4}$

Let  $N_2$  be the last  $n > N_0$  with  $f(n) - f(N_0) < \frac{1}{2}$

Let  $N_3$  be the last  $n > N_0$  with  $f(n) - f(N_0) < \frac{3}{4}$

Let  $N_4$  be the last  $n > N_0$  with  $f(n) - f(N_0) < 1$

Let  $d = \frac{3}{2\pi}$ . Note: The choice of the number 3 on the numerator of  $d$  was an arbitrary choice, we could just have easily have chosen any number less than  $\pi$ . We will now define  $M_i$ 's,  $i = 0, \dots, 3$  so that  $|f(M_1) - f(M_0)| \simeq d$  and  $|f(M_3) - f(M_2)| \simeq d$ . The intervals  $[f(M_0), f(M_1)]$  and  $[f(M_2), f(M_3)]$  will be placed approximately in the center of the intervals  $[0, \frac{1}{2}]$  and  $[\frac{1}{2}, 1]$  respectively.

Let  $M_0$  be the last  $n > N_0$  with  $f(n) - f(N_0) < \frac{1}{2}(\frac{1}{2} - d)$

Let  $M_1$  be the first  $n > N_0$  with  $f(n) - f(N_0) > \frac{1}{2}(\frac{1}{2} + d)$

Let  $M_2$  be the last  $n > N_0$  with  $f(n) - f(N_0) < \frac{3}{2}(\frac{3}{2} - d)$

Let  $M_3$  be the first  $n > N_0$  with  $f(n) - f(N_0) > \frac{3}{2}(\frac{3}{2} + d)$

We will show that

$$\Im \sum_{n=N_0}^{N_4} e^{2\pi i f(n)} < 0 \quad (3.8)$$

and

$$\frac{1}{M_3} \Im \sum_{n=N_0}^{M_3} e^{2\pi i f(n)} < -0.01 \quad (3.9)$$

and that (3.8) and (3.9) combine to give

$$\lim_{N \rightarrow \infty} \frac{1}{N} \Im \sum_{n=1}^N e^{2\pi i f(n)} \neq 0 \quad (3.10)$$

so (2.2) fails and hence  $(f(n))$  is not uniformly distributed modulo 1.

We will need the following lemma

**Lemma 3.5.2** *let  $f$  be a function whcih satisfies the hypothesis of Theorem 3.5.1. Then:*

$$f' \left( f^{-1} \left( f(a) + \frac{1}{2} \right) \right) < \frac{1}{3} f'(a)$$

Note: Since  $f'(x) > 0 \forall x$  then  $f$  has an inverse.

**Proof**

By the MVT and our assumptions about  $f'(x)$ ,

$$\begin{aligned}
 f^{-1}\left(f(a) + \frac{1}{2}\right) &\geq \left|f^{-1}\left(f(a) + \frac{1}{2}\right) - a\right| \\
 &= \frac{\left|f(a) + \frac{1}{2} - f(a)\right|}{f'(\xi)} \quad (\text{where } a \leq \xi \leq f^{-1}\left(f(a) + \frac{1}{2}\right)) \\
 &\geq \frac{1}{2f'(a)} \\
 f'\left(f^{-1}\left(f(a) + \frac{1}{2}\right)\right) &\leq \frac{1}{f^{-1}\left(f(a) + \frac{1}{2}\right)} \\
 &\leq \frac{2f'(a)}{2\pi} \\
 &< \frac{1}{3}f'(a)
 \end{aligned}$$

■

**Corollary 3.5.3** *If we let  $a_j = N_0 + j$ ,  $j = 0, 1, \dots, M_1 - N_0 - 1$ , then for each  $j$  there are at least three points,  $a_{0j}$ ,  $a_{1j}$ ,  $a_{2j}$  which satisfy  $a_{ij} \in [N_2 + 1, M_3]$  and  $f(a_j) + \frac{1}{2} \leq f(a_{ij}) \leq f(a_j + 1) + \frac{1}{2}$ ,  $i = 0, 1, 2$ .*

### 3.5.2 Dealing with terms lying between $N_0$ and $N_1 - 1$

Let  $m = f'(N_2)$ , Then for  $n > N_2$ ,  $|f(n) - f(n-1)| < m$  so,  $\forall j > 0$   $\exists S_j > N_2$  such that  $(f(S_j) - f(N_0)) \in [\frac{1}{2} + jm, \frac{1}{2} + (j+1)m)$ .

Also, for  $N_0 \leq n \leq N_1$   $|f(n) - f(n-1)| \geq m$ , so  $\forall j \geq 1$ ,  $k = 1, \dots, (N_1 - N_0 - 1)$  if  $(f(N_0 + k) - f(N_0)) \in [(j-1)m, jm)$  then it is the only  $f(N_0 + k)$  which has this property.

Now let  $j_k$  be the  $j \geq 1$  such that  $(f(N_0 + k) - f(N_0)) \in [(j_k - 1)m, j_k m)$  and let  $S_k$  satisfy  $N_2 < S_k \leq N_3$  and  $(f(S_k) - f(N_0)) \in [\frac{1}{2} + j_k m, \frac{1}{2} + (j_k + 1)m)$ .

Remember that  $\sin 2\pi(f(n) - f(N_0)) = \sin 2\pi f(n)$  and in addition  $\sin 2\pi f(N_0 + k) < |\sin 2\pi f(S_k)|$  so

$$\begin{aligned}
 \sum_{n=N_0+1}^{N_1-1} \sin 2\pi f(n) &= \sum_{k=1}^{N_1-N_0-1} \sin 2\pi f(N_0 + k) \\
 &< \sum_{k=1}^{N_1-N_0-1} |\sin 2\pi f(S_k)|
 \end{aligned} \tag{3.11}$$



### 3.5.3 Dealing with terms lying between $N_1 + 2$ and $N_2$

Let  $M = f'(N_3)$ , Then for  $n > N_3$ ,  $|f(n) - f(n-1)| < M$  so,  $\forall j > 0 \exists R_j$ ,  $R_j > N_3$  such that  $(f(R_j) - f(N_0)) \in [1 - (j+1)M, 1 - jM)$

Also, for  $N_1 \leq n \leq N_2$   $|f(n) - f(n-1)| \geq M$ , so  $\forall j \geq 1, k = 0, \dots, (N_2 - N_1 - 2)$  if  $(f(N_2 - k) - f(N_0)) \in [\pi - jM, \pi - (j-1)M)$  then it is the only  $f(N_2 - k)$  which has this property.

Now let  $j_k$  be the  $j \geq 1$  such that  $(f(N_2 - k) - f(N_0)) \in [\frac{1}{2} - j_k M, \frac{1}{2} - (j_k - 1)M)$ . Starting with  $k = N_2 - N_1 - 2$ , and decrementing the value of  $k$  by 1 each time let  $R_k$  be chosen as follows:

If  $(1 - j_k M) < \frac{1}{2}(\frac{3}{2} + d)$  then pick  $R_k$  so that  $(f(R_k) - f(N_0)) \in [1 - (j_k + 1)M, 1 - j_k M)$ . Otherwise pick  $R_k$  with  $M_2 < R_k < M_3$  and  $R_k \neq R_j$  for  $j > k$ ,  $R_k \neq S_i$  for any  $i$ .

Then  $\sin 2\pi f(N_2 - k) < |\sin 2\pi f(R_k)|$  and

$$\begin{aligned} \sum_{n=N_1+2}^{N_2} \sin 2\pi f(n) &= \sum_{k=0}^{N_2-N_1-2} \sin 2\pi f(N_2 - k) \\ &< \sum_{k=0}^{N_2-N_1-2} |\sin 2\pi f(R_k)| \end{aligned} \quad (3.12)$$

### 3.5.4 Dealing with the terms $N_1$ and $N_1 + 1$

By Corollary 3.5.3 there are at least three terms  $a_i$ ,  $i = 0, 1, 2$  with  $f(a_i) \in [\frac{1}{2} + f(N_1), \frac{1}{2} + f(N_1 + 1)]$ , and these  $a_i$ 's do not correspond to any  $S_k$  or  $R_k$ . Without loss of generality assume  $a_0 < a_1 < a_2$ .

We need to show that

$$\sum_{i=0}^2 |\sin 2\pi f(a_i)| > \sin 2\pi f(N_1) + \sin 2\pi f(N_1 + 1). \quad (3.13)$$

If  $f(a_0) \leq \frac{3\pi}{2}$  and  $f(a_2) \geq \frac{3\pi}{2}$  then (3.13) clearly holds.

We now consider the cases where  $(f(a_0) - f(N_0)) > \frac{3}{4}$  or  $(f(a_2) - f(N_0)) < \frac{3}{4}$ . These can essentially be treated as the same case, so w.l.o.g. assume  $f(a_0) > \frac{3}{4}$ . Then  $|\sin 2\pi f(a_2)| \geq \sin 2\pi f(N_1 + 1)$  and it remains to show that  $|\sin 2\pi f(a_0)| + |\sin 2\pi f(a_1)| \geq \sin 2\pi f(N_1)$ .

Let  $D = f(N_1 + 1) - f(N_1)$ . Then  $f(a_1) - f(a_0) \leq \frac{D}{3}$  and  $f(a_2) - f(a_1) \leq \frac{D}{3}$ . Now we know that  $f(N_1 + 1) - f(N_1) < \frac{1}{6N_1}$ , however we will assume that  $(f(N_1 + 1) - f(N_1)) \sim \frac{1}{2}$  and thus  $f(a_0) \simeq \frac{5}{6}$  and  $f(a_1) \simeq \frac{11}{12}$  as this is

the worst possible scenario. Even in this worst case scenario

$$\begin{aligned} |\sin 2\pi f(a_0)| + |\sin 2\pi f(a_1)| &\geq \frac{\sqrt{3}}{2} + \frac{1}{2} \\ &> 1 \\ &> \sin 2\pi f(N_1) \end{aligned}$$

and so (3.13) holds.

### 3.5.5 Combining the results

Combining (3.11), (3.12) and (3.13) we get:

$$\begin{aligned} \sum_{n=N_0}^{N_2} \sin 2\pi f(n) &= \sum_{\substack{n=N_0+1 \\ N_1-N_0-1}}^{N_1-1} \sin 2\pi f(n) + \sum_{n=N_1+1}^{N_2} \sin 2\pi f(n) + \sin 2\pi f(N_1) + \sin 2\pi f(N_1+1) \\ &< \sum_{k=1}^{N_1-N_0-1} |\sin 2\pi f(S_k)| + \sum_{k=1}^{N_1-N_0-1} |\sin 2\pi f(R_k)| + \sum_{i=0}^2 |\sin 2\pi f(a_i)| \end{aligned}$$

and so

$$\sum_{n=N_0}^{N_2} \sin 2\pi f(n) + \sum_{k=1}^{N_1-N_0-1} \sin 2\pi f(S_k) + \sum_{k=1}^{N_1-N_0-1} \sin 2\pi f(R_k) + \sum_{i=0}^2 \sin 2\pi f(a_i) < 0 \quad (3.14)$$

Now since  $\Im e^{2\pi i f(n)} < 0$  for  $M_3 \leq n \leq N_4$  then

$$\begin{aligned} \Im \sum_{n=N_0}^{N_4} e^{2\pi i f(n)} &\leq \Im \sum_{n=N_0}^{M_3} e^{2\pi i f(n)} \\ &= \sum_{n=N_0}^{N_2} \sin 2\pi f(n) + \sum_{\substack{n=N_2+1 \\ N_1-N_0-1}}^{N_3} \sin 2\pi f(n) \\ &\leq \sum_{n=N_0}^{N_2} \sin 2\pi f(n) + \sum_{k=1}^{N_1-N_0-1} \sin 2\pi f(S_k) \\ &\quad + \sum_{k=1}^{N_1-N_0-1} \sin 2\pi f(R_k) + \sum_{i=0}^2 \sin 2\pi f(a_i) \\ &< 0 \quad (\text{by (3.14)}) \end{aligned}$$

Thus (3.8) holds.

### 3.5.6 Using the remaining terms

We now let  $(m_i)$  be the terms with  $M_2 < m_i \leq M_3, i = 1, 2, \dots, M_3 - M_2$  and ask what fraction of the  $m_i$ 's do not correspond to an  $S_k$  or  $R_k$ . No  $m_i$ 's were used to compensate for terms with  $N_0 \leq n \leq M_0$ . By the corollary, at most  $\frac{1}{3}$  of the  $m_i$ 's were used to compensate for terms with  $M_0 < n \leq M_1$ . To compensate for terms with  $M_1 < n \leq N_2$  another  $N_2 - M_1$  of the  $a_i$ 's were used

Using the M.V.T. and our assumptions about  $f'(x)$  and  $f''(x)$  we see that

$$\begin{aligned} |M_3 - M_2| &\geq \frac{1}{f'(M_2)} |f(M_3) - f(M_2)| \\ &\geq 2\pi M_2 d \\ &\geq 3M_2 \end{aligned} \tag{3.15}$$

Using (3.15) we get

$$\begin{aligned} \frac{1}{3} &\geq \frac{M_2}{M_3 - M_2} \\ &\geq \frac{M_2 - M_1}{M_3 - M_2} \\ &\geq \frac{M_3 - M_2}{N_2 - M_1} \\ &\geq \frac{N_2 - M_1}{M_3 - M_2} \end{aligned}$$

and thus at most another  $\frac{1}{3}$  of the  $a_i$ 's were used up here. Hence there are at least  $\frac{1}{3}$  of the  $a_i$ 's left which do not correspond to an  $S_k$  or  $R_k$ .

Hence

$$\begin{aligned} \frac{1}{M_3} \sum_{n=N_0}^{M_3} \sin 2\pi f(n) &= \frac{1}{M_3} \left( \sum_{N_0}^{N_2} \sin 2\pi f(n) + \sum_{k=1}^{N_1 - N_0 - 1} \sin 2\pi f(S_k) \right. \\ &\quad \left. + \sum_{k=1}^{N_2 - N_1 - 2} \sin 2\pi f(R_k) + \sum_{i=0}^2 \sin 2\pi f(a_i) \right) \\ &\quad + \frac{1}{M_3} \sum_{n=N_2+1, n \neq S_k, R_k, a_i}^{M_3} \sin 2\pi f(n) \\ &< \frac{1}{M_3} \sum_{n=M_2, n \neq S_k, R_k, a_i}^{M_3} \sin 2\pi f(n) \end{aligned} \tag{3.16}$$

Now for  $n$ 's appearing in the last term of (3.16)  $\sin 2\pi f(n) < 0$ . Also, for  $M_2 < n < M_3$   $|\sin 2\pi f(n)| \geq 0.07$ . Finally using (3.15) again we can deduce that  $\frac{M_2}{M_3} \leq \frac{1}{4}$ . Using these facts and (3.16) we get:

$$\begin{aligned}
-\frac{1}{M_3} \sum_{n=M_2, n \neq S_k, R_k, a_i}^{M_3} \sin 2\pi f(n) &= \frac{1}{M_3} \sum_{n=M_2, n \neq S_k, R_k, a_i}^{M_3} |\sin 2\pi f(n)| \\
&> \frac{1}{M_3} \frac{0.07}{3} (M_3 - M_2) \\
&> 0.02 \left(1 - \frac{M_2}{M_3}\right) \quad (\text{where } C \text{ is a constant}) \\
&> 0.02 \left(1 - \frac{1}{4}\right) \\
&> 0.01
\end{aligned} \tag{3.17}$$

Thus combining (3.16) and (3.17) gives (3.9).

### 3.5.7 Deriving (3.10)

We now slightly change the notation to let  $N_0 = N_{00} = f(1)$ , and let  $N_{i0} = N_i$  and  $M_{i0} = M_i$ . Now define  $N_{01} = N_{40} + 1$  and take  $N_{i1}$  and  $M_{i1}$  to be the  $N_i$ 's and  $M_i$ 's which we would have gotten if we took  $N_0 = N_{01}$ . Continue on in this manner, so that we get  $N_{ij}$  to be the  $N_i$  obtained on the " $j$ th" revolution around the circle. Then

$$\begin{aligned}
\frac{1}{M_{3j}} \Im \sum_{n=1}^{M_{3j}} e^{2\pi i f(n)} &= \frac{1}{M_{3j}} \sum_{n=1}^{M_{3j}} \sin 2\pi f(n) \\
&= \frac{1}{M_{3j}} \sum_{k=0}^{j-1} \underbrace{\sum_{N_{0k}}^{N_{4k}} \sin 2\pi f(n)}_{<0} + \frac{1}{M_{3j}} \underbrace{\sum_{n=N_{0j}}^{M_{3j}} \sin 2\pi f(n)}_{<-0.01} \quad (3.18) \\
&< -0.01
\end{aligned}$$

(3.18) shows that as  $N$  increases

$$\frac{1}{N} \Im \sum_{n=1}^N e^{2\pi i f(n)} < -0.01$$

whenever  $N = M_{3j}$ . This happens infinitely often and thus (3.10) holds. Hence the sequence  $(f(n))$  is not uniformly distributed modulo 1. ■

### 3.6 The Distribution of $u_n + f(n)$

**Theorem 3.6.1** *If  $(u_n)$  is an arbitrary sequence which is uniformly distributed modulo 1 and  $|f'(x)| < \frac{K}{x}$  then  $(u_n + f(n))$  is uniformly distributed modulo 1.*

**Proof**

In this proof we will use Weyl's criterion as well as the formula below:

$$\sum_{n=m}^N b_n(a_n - a_{n-1}) = b_{N+1}a_N - b_m a_{m-1} - \sum_{n=m}^N (b_{n+1} - b_n)a_n \quad (3.19)$$

This technique is known as summation by parts, and can be verified by expanding both sides of (3.19).

Now let

$$a_m = \sum_{j=1}^m e^{2\pi i u_j} \text{ and } b_m = e^{2\pi i f(m)}$$

Then

$$\begin{aligned} \sum_{n=2}^N e^{2\pi i h(u_n + f(n))} &= \sum_{n=2}^N e^{2\pi i h u_n} e^{2\pi i h f(n)} \\ &= e^{2\pi i h f(N+1)} \sum_{j=1}^N e^{2\pi i h u_j} - e^{2\pi i h (u_1 + f(2))} \\ &\quad - \underbrace{\sum_{n=2}^N \left( (e^{2\pi i h f(n+1)} - e^{2\pi i h f(n)}) \sum_{j=1}^n e^{2\pi i h u_j} \right)}_S \quad (\text{using (3.19)}) \end{aligned} \quad (3.20)$$

Dividing both sides of (3.20) by  $N$  and taking the lim sup as  $N \rightarrow \infty$  we see that the first and second terms tend to 0.

We now consider  $|e^{2\pi i h f(n)} - e^{2\pi i h f(n-1)}|$ .

Let  $g(x) = e^{2\pi i h f(x)}$ . Then  $|g'(x)| \leq \frac{K}{x}$  and for  $n \leq x \leq n+1$ ,  $|g'(x)| \leq \frac{K}{n} \Rightarrow |g(n+1) - g(n)| \leq \frac{K}{n}$ .

That is

$$|e^{2\pi i h f(n)} - e^{2\pi i h f(n-1)}| \leq \frac{K}{n} \quad (3.21)$$

We now consider  $S$  from (3.20)

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|S|}{N} &= \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=2}^N (e^{2\pi i h f(n+1)} - e^{2\pi i h f(n)}) \sum_{j=1}^n e^{2\pi i h u_j} \right| \\
&\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=2}^N |e^{2\pi i h f(n+1)} - e^{2\pi i h f(n)}| \left| \sum_{j=1}^n e^{2\pi i h u_j} \right| \\
&\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=2}^N \frac{K}{n} \left| \sum_{j=1}^n e^{2\pi i h u_j} \right| \quad (\text{using (3.21)}) \\
&= \limsup_{N \rightarrow \infty} \frac{1}{N} \frac{K}{N} \sum_{n=2}^N \left| \frac{1}{n} \sum_{j=1}^n e^{2\pi i h u_j} \right|
\end{aligned}$$

Now  $(u_n)$  is u.d., so

$$\left| \frac{1}{n} \sum_{j=1}^n e^{2\pi i h u_j} \right| \rightarrow 0 \text{ as } n \rightarrow \infty$$

and hence for each  $\epsilon$  there is a  $K_\epsilon$  such that for  $n > K_\epsilon$ ,

$$\left| \frac{1}{n-1} \sum_{j=1}^n e^{2\pi i h u_j} \right| < \epsilon.$$

Hence

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|S|}{N} &= \limsup_{N \rightarrow \infty} \frac{K}{N} \left( \sum_{n=2}^{K_\epsilon} \left| \frac{1}{n-1} \sum_{j=1}^n e^{2\pi i h u_j} \right| + \sum_{n=K_\epsilon+1}^N \left| \frac{1}{n-1} \sum_{j=1}^n e^{2\pi i h u_j} \right| \right) \\
&\leq \limsup_{N \rightarrow \infty} \frac{K}{N} \left( K'_\epsilon + \sum_{n=K_\epsilon+1}^N \epsilon \right) \\
&= \limsup_{N \rightarrow \infty} \frac{K}{N} (K'_\epsilon + (N - K_\epsilon - 1)\epsilon) \\
&\leq \epsilon
\end{aligned}$$

Since we can pick  $\epsilon$  to be arbitrarily small we conclude that the term  $S$  in (3.20) tends to zero as well. This tells us that Weyl's criterion holds for the sequence  $(u_n + f(n))$  this sequence is uniformly distributed modulo 1. ■

As a corollary to theorem 3.6.1 we get

**Theorem 3.6.2** *If  $|f'(x)| \leq \frac{K}{x}$  then  $(f(n))$  is not uniformly distributed modulo 1.*

This is a much stronger statement than Theorem 3.5.1.

**Proof**

Assume that  $(f(n))$  was u.d. mod 1. Then  $(-f(n))$  would also be u.d. mod 1 and hence letting  $u_n = -f(n)$  we would arrive at the conclusion that 0 is u.d. mod 1, which is clearly false. Hence  $(f(n))$  cannot be uniformly distributed modulo 1. ■

**Corollary 3.6.3**  *$(\log n)$  is not uniformly distributed modulo 1.*

Note: This corollary was already proven by other methods as Proposition 3.2.1.

**Proof**

$\frac{d}{dx} \log x = \frac{1}{x}$  and so theorem 3.6.2 can be applied to  $f(x) = \log x$  to yield the desired result. ■

## Chapter 4

# Trigonometric Sequences

In this chapter we examine the distribution of various families of trigonometric sequences. Many of the arguments in chapter 3 relied on the fact that the sequences we were considering had a monotonic derivative, however this is no longer the case. As a result we have to employ slightly different techniques to analyse the distribution of these sequences. The chapter begins by analysing the most basic family of trigonometric sequences,  $K \sin x_n$  (where  $x_n$  is u.d. mod  $2\pi$ ) and builds up to analyse more complicated families of sequences. The results combine to give a comprehensive analysis of sequences of the form  $(n^\alpha \sin n^\beta)$  for  $\beta < 1$ , and also include some results for more general families,  $(f(n) \sin n^\beta)$ ,  $\beta < 1$  and  $(n^\alpha \sin n)$ ,  $\alpha < \frac{1}{2}$ .

As a general rule trigonometric sequences are difficult to analyse as their derivative varies wildly, making them hard to integrate. By looking at sequences whose trigonometric part is a function only of  $n^\beta$ ,  $\beta < 1$  we at least get to tame this derivative as  $n$  gets very large and so the analysis is easier.

As far as we know, there has been no analysis of sequences of the form  $(n^\alpha \sin n^\beta)$  for  $\beta > 1$ .



## 4.1 The distribution of $(K \sin(x_n))$

**Theorem 4.1.1** *For  $K > 1$  and  $(x_n)$  u.d. mod  $2\pi$ ,  $(K \sin(x_n))$  is not uniformly distributed modulo 1.*

**Proof**

Assume  $(K \sin x_n)$  is u.d. mod 1. Pick  $0 < \epsilon < \frac{1}{2K^3\pi^2}$ . Now  $\{K \sin x_n\} \in (\{K\}, \{K\} + \epsilon)$  iff

$$\begin{aligned} \{x_n\}_{2\pi} \in & \bigcup_{i=0}^{\lfloor K-1 \rfloor} \left( \left( \sin^{-1} \frac{i + \{K\}}{K}, \sin^{-1} \frac{i + \{K\} + \epsilon}{K} \right) \cup \right. \\ & \left. \left( \pi - \sin^{-1} \frac{i + \{K\} + \epsilon}{K}, \pi - \sin^{-1} \frac{i + \{K\}}{K} \right) \right) \\ & \bigcup_{i=1}^{\lfloor K \rfloor} \left( \left( \pi + \sin^{-1} \frac{i + \{K\} - \epsilon}{K}, \pi + \sin^{-1} \frac{i + \{K\}}{K} \right) \cup \right. \\ & \left. \left( 2\pi - \sin^{-1} \frac{i + \{K\}}{K}, 2\pi - \sin^{-1} \frac{i + \{K\} - \epsilon}{K} \right) \right) \end{aligned}$$

and thus

$$\begin{aligned} \phi_N^{K \sin x_n}(\{K\}, \{K\} + \epsilon) &= 2 \sum_{i=0}^{\lfloor K-1 \rfloor} \phi_N^{x_n} \left( \sin^{-1} \frac{i + \{K\}}{K}, \sin^{-1} \frac{i + \{K\} + \epsilon}{K} \right) \\ &\quad + 2 \sum_{i=i}^{\lfloor K \rfloor} \phi_N^{x_n} \left( \sin^{-1} \frac{i + \{K\} - \epsilon}{K}, \sin^{-1} \frac{i + \{K\}}{K} \right) \\ &> 2 \phi_N^{x_n} \left( \sin^{-1} \frac{K - \epsilon}{K}, \frac{\pi}{2} \right) \\ &\quad + 2 \sum_{i=1}^{\lfloor K-1 \rfloor} \phi_N^{x_n} \left( \sin^{-1} \frac{i + \{K\} - \epsilon}{K}, \sin^{-1} \frac{i + \{K\} + \epsilon}{K} \right) \end{aligned} \tag{4.1}$$

Since  $(x_n)$  is uniformly distributed modulo  $2\pi$  we know that  $\lim_{N \rightarrow \infty} \frac{\phi_N^{x_n}([a, b])}{N} = \frac{b-a}{2\pi}$ . If we divide both sides of (4.1) by  $N$  and try to take limits as  $N \rightarrow \infty$  we see that all the limits on the right hand side exist and thus so must the limit

on the left hand side. Hence we get:

$$\begin{aligned}
\lim_{N \rightarrow \infty} \frac{1}{N} \phi_N^{K \sin x_n}(\{K\}, \{K\} + \epsilon) &> \frac{1}{\pi} \left( \left( \frac{\pi}{2} - \sin^{-1} \frac{K - \epsilon}{K} \right) \right. \\
&\quad \left. + \sum_{i=1}^{K-1} \left( \sin^{-1} \frac{i + \{K\} + \epsilon}{K} - \sin^{-1} \frac{i + \{K\} - \epsilon}{K} \right) \right) \\
&> \frac{1}{\pi} \left( \frac{\pi}{2} - \sin^{-1} \frac{K - \epsilon}{K} \right) \\
&> \frac{\epsilon}{\pi K} \frac{d}{dx} \left( \sin^{-1} \frac{K - \epsilon}{K} \right) \\
&\quad \text{(By the MVT)} \\
&= \frac{\epsilon}{\pi K} \frac{1}{\sqrt{K^2 - (K - \epsilon)^2}} \\
&= \frac{\epsilon}{\pi K} \frac{1}{\sqrt{2K\epsilon - \epsilon^2}} \\
&> \frac{\epsilon}{\pi K^{\frac{3}{2}} \sqrt{2\epsilon}}
\end{aligned} \tag{4.2}$$

However if  $(K \sin x_n)$  is u.d. mod 1 then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \phi_N^{K \sin x_n}(\{K\}, \{K\} + \epsilon) = \epsilon. \tag{4.3}$$

Thus (4.2) and (4.3) combine to give:

$$\pi > \frac{1}{K^{\frac{3}{2}} \sqrt{2\epsilon}} \tag{4.4}$$

but

$$\begin{aligned}
\frac{1}{K^{\frac{3}{2}} \sqrt{2\epsilon}} &> \frac{1}{K \sqrt{\frac{2K}{2K^3 \pi^2}}} \\
&= \pi
\end{aligned} \tag{4.5}$$

Which is a contradiction. Thus for  $K > 1$   $(K \sin x_n)$  cannot be uniformly distributed modulo 1. ■

The following theorem follows as a corollary from Theorem 4.1.1.

**Theorem 4.1.2**  $(\sin n^\beta)$  is not uniformly distributed modulo 1.

**Proof**

The proof of this Theorem breaks into three cases.

Case 1:  $\beta > 0$

Proposition 3.3.1 proved that  $(2\pi n^\beta)$  is uniformly distributed modulo 1. Thus  $(n^\beta)$  is uniformly distributed modulo  $2\pi$  and the result follows from Theorem 4.1.1.

Case 2:  $\beta = 0$

This case is trivial as we are talking about a constant sequence which cannot be u.d. mod 1.

Case 3:  $\beta < 0$

In this case  $\lim_{n \rightarrow \infty} \sin n^\beta = 0$  and so all the terms of the sequence are positive and tending towards zero and hence their fractional parts are also tending to zero. Thus the sequence cannot be u.d. mod 1. ■

## 4.2 The distribution of $n^{-\alpha} \sin n^\beta$

**Theorem 4.2.1** *If  $\alpha > 0$  then  $(n^{-\alpha} \sin n^\beta)$  is not uniformly distributed modulo 1.*

**Proof**

In this case  $\lim_{n \rightarrow \infty} n^{-\alpha} \sin n^\beta = 0$  and so all the terms of the sequence are tending towards zero. This means that their fractional parts are either very close to zero or very close to one. Thus the fractional parts cannot be uniformly distributed in the unit interval and so  $(n^{-\alpha} \sin n^\beta)$  will not be u.d. mod 1. ■

## 4.3 The distribution of $n^\alpha \sin n^{-\beta}$

**Theorem 4.3.1** *For  $\beta > 0$*

- (i) *If  $\alpha > \beta$  and  $(\alpha - \beta) \notin \mathbb{Z}$   $(n^\alpha \sin n^{-\beta})$  is uniformly distributed modulo 1.*
- (ii) *If  $\alpha > \beta$  and  $(\alpha - \beta) \in \mathbb{Z}$  Then  $(n^\alpha \sin n^{-\beta})$  is uniformly distributed modulo 1 iff  $\beta$  is not a multiple of  $\frac{1}{2}$ .*
- (iii) *If  $\alpha \leq \beta$   $(n^\alpha \sin n^{-\beta})$  is not uniformly distributed modulo 1.*

**Proof**

We consider the “Taylor” series expansion of  $n^\alpha \sin n^{-\beta}$ . We know that

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (4.6)$$

Substituting  $x = n^{-\beta}$  into (4.6) and multiplying by  $n^\alpha$  we get:

$$\begin{aligned}
n^\alpha \sin n^{-\beta} &= \sum_{k=0}^{\infty} (-1)^k \frac{n^{\alpha-\beta(2k+1)}}{(2k+1)!} \\
&= n^{\alpha-\beta} \sum_{k=0}^{\infty} (-1)^k \frac{n^{-2k\beta}}{(2k+1)!} \\
&= n^{\alpha-\beta} + n^{\alpha-\beta} \sum_{k=1}^{\infty} (-1)^k \frac{n^{-2k\beta}}{(2k+1)!}
\end{aligned} \tag{4.7}$$

(i) Define  $\gamma = (\alpha - \beta)$ . Then  $\gamma > 0$  and  $\gamma$  is not an integer. Let  $T = \lfloor \frac{\gamma}{2\beta} \rfloor$ . We can re-write (4.7) as

$$n^\alpha \sin n^{-\beta} = \underbrace{n^\gamma + \sum_{k=1}^T (-1)^k \frac{n^{\gamma-2k\beta}}{(2k+1)!}}_{f(n)} + \underbrace{\sum_{k=T+1}^{\infty} (-1)^k \frac{n^{\gamma-2k\beta}}{(2k+1)!}}_{\epsilon_n} \tag{4.8}$$

Then  $f(x)$  satisfies the hypothesis for Proposition 3.3.1 and is thus uniformly distributed modulo 1. In addition  $\epsilon_n \rightarrow 0$  as  $N \rightarrow \infty$  and so using Corollary 2.2.4 we can conclude that  $(n^\alpha \sin n^{-\beta})$  is uniformly distributed modulo 1 for the conditions given in part (i).

(ii) Define  $m = (\alpha - \beta)$ . Then  $m > 0$  and  $m \in \mathbb{Z}$ . Let  $T = \lfloor \frac{m}{2\beta} \rfloor$ . Thus (4.7) can be written as:

$$n^\alpha \sin n^{-\beta} = n^m + \underbrace{n^{m-2\beta} + \sum_{k=2}^T (-1)^k \frac{n^{m-2k\beta}}{(2k+1)!}}_{f(n)} + \underbrace{\sum_{k=T+1}^{\infty} (-1)^k \frac{n^{\gamma-2k\beta}}{(2k+1)!}}_{\epsilon_n} \tag{4.9}$$

where  $\epsilon_n \rightarrow 0$  as  $N \rightarrow \infty$ . Since the fractional part of  $n^m$  is always zero the  $n^m$  term in (4.9) does not affect the distribution of the sequence. Thus  $(n^\alpha \sin n^{-\beta})$  is distributed in the same manner as  $(f(n) + \epsilon_n)$ . We now have two cases.

Case 1:  $\beta$  is not a multiple of  $\frac{1}{2}$ .

The leading term of  $f(n)$  is  $n^{m-2\beta}$ . Since  $\beta$  is not a multiple of  $\frac{1}{2}$  then  $(m-2\beta) \notin \mathbb{Z}$  and so  $f(n)$  obeys the hypothesis of Proposition 3.3.1 and so is u.d. mod 1. Using this fact and Corollary 2.2.4 we conclude that the reverse direction of (ii) holds.

Case 2:  $\beta$  is a multiple of  $\frac{1}{2}$ .

In this case the powers of  $n$  in  $f(n)$  are of the form  $m - 2k\beta \in \mathbb{Z}$  and so

$f(n)$  is a polynomial with no irrational coefficients and by Proposition 3.1.3 is not uniformly distributed modulo 1. Combining this with Corollary 2.2.4 we conclude that the forward direction of (ii) holds.

(iii) In this case we have  $\alpha \leq \beta$ , and so  $n^{\alpha-\beta} \leq 1$ . In addition the terms in the summation of 4.7 are all approaching zero as  $n$  increases. Pick  $\epsilon > 0$ . Then for large enough  $n$  the second term on the right hand side of (4.7) less than  $\epsilon$ . We are now left with two cases. Either  $\alpha = \beta$  and we get

$$n^\alpha \sin n^{-\beta} = 1 + O(\epsilon) \quad (4.10)$$

or  $\alpha < \beta$  and choosing  $n$  large enough we can make  $n^{\alpha-\beta} < \epsilon$  and thus

$$n^\alpha \sin n^{-\beta} = O(\epsilon) \quad (4.11)$$

Either way  $(n^\alpha \sin n^{-\beta})$  cannot be uniformly distributed modulo 1 as it's fractional parts are all  $O(\epsilon)$ . Thus part (iii) of the theorem is proved. ■

## 4.4 The distribution of $n^\alpha \sin n^\beta$

The main result of this section is contained in Theorem 4.4.9 and states that for  $\beta < 1$  and  $\alpha > 0$  then  $n^\alpha \sin n^\beta$  is uniformly distributed modulo 1. Different methods are needed to analyse the sequence depending on whether  $\alpha$  is a multiple of  $1 - \beta$  or not. For this reason we break the discussion up into two parts. The first part (section 4.4.1) deals with the case when  $\alpha$  is not a multiple of  $1 - \beta$  and the second part (section 4.4.2) deals with sequences where  $\alpha$  is a multiple of  $1 - \beta$ .

Throughout this section we will make repeated use of the following two lemmas.

**Lemma 4.4.1** *If  $f(x)$  and  $g(x)$  are increasing functions with  $g'(x) < K$  and  $A_{M\epsilon}$  is the set defined by  $A_{M\epsilon} = \{x \mid \epsilon M \leq x \leq M, |\cos g(x)| \geq \epsilon\}$  then*

$$|[f(x)]_{A_{M\epsilon}}| \leq K f(M) g(M)$$

### Proof

We begin by observing that  $A_{M\epsilon}$  is broken up into  $T$  intervals, where  $T$  is the number of times that  $\cos g(x)$  crosses the x-axis. Let us denote these

intervals by  $I_t = (a_t, b_t)$  where  $t = 0 \dots T$ . Then certainly

$$\begin{aligned} |[f(x)]_{A_\epsilon}| &= \left| \sum_{t=1}^T [f(x)]_{I_t} \right| \\ &\leq \sum_{t=1}^T |f(b_t) - f(a_t)| \\ &\leq 2Tf(M) \end{aligned} \tag{4.12}$$

(Since  $f$  is an increasing function.)

It remains for us to show how many times  $\cos g(x)$  crosses the x-axis. If  $\cos g(x) = 0$  then  $g(x) = (2t+1)\frac{\pi}{2}$ ,  $t \in \mathbb{Z}$ . In addition,  $x \leq M$  so we must have

$$g^{-1}\left((2t+1)\frac{\pi}{2}\right) < M \tag{4.13}$$

Note that since  $g$  is increasing,  $g^{-1}$  exists. It is easy to solve (4.13) to get  $t \leq Kg(M)$ . Thus we can conclude that  $T \leq Kg(M)$  and thus the lemma is proved.  $\blacksquare$

**Lemma 4.4.2** *If  $f$ ,  $A_{M\epsilon}$  are defined as in Lemma 4.4.1 and  $g(x) = x^\beta + \phi$  for  $\beta < 1$  and  $B_{M\epsilon} = [\epsilon M, M] \setminus A_{M\epsilon}$  then*

$$\limsup_{M \rightarrow \infty} \frac{1}{M} \left| \int_{B_{M\epsilon}} e^{2\pi i h f(x)} dx \right| = O(\epsilon)$$

**Proof**

We can define  $B_{M\epsilon}$  alternatively as  $B_{M\epsilon} = \{x \mid \epsilon M \leq x \leq M, |\cos x^\beta + \phi| < \epsilon\}$ . As was the case with  $A_{M\epsilon}$  this set is divided up into  $T$  intervals where once again  $T$  is the number of times which  $\cos(x^\beta + \phi)$  crosses the x-axis. From above we have that  $T \leq KM^\beta$ . We now ask what the length of each one of these intervals is. This time we denote the intervals of  $B_{M\epsilon}$  by  $J_t = (a_t, b_t)$ .

It is not too hard to see that  $|b_t^\beta - a_t^\beta| \leq 2 \sin^{-1} \epsilon$  and thus  $a_t \geq ((b_t)^\beta - 2 \sin^{-1} \epsilon)^{\frac{1}{\beta}}$ . Thus the length of an interval  $J_t$  of  $B_{M\epsilon}$  is given by:

$$\begin{aligned} |J_t| &\leq b_t - ((b_t)^\beta - 2 \sin^{-1} \epsilon)^{\frac{1}{\beta}} \\ &\sim (b_t)^\beta \sin^{-1} \epsilon + l.o.t. \\ &\quad \text{(Using the M.V.T. since } b_t^\beta \gg \sin^{-1} \epsilon) \\ &\leq K(M^{1-\beta} \sin^{-1} \epsilon + l.o.t.) \\ &\quad \text{(Since } x^\beta \text{ is an increasing function)} \end{aligned} \tag{4.14}$$

Thus we get

$$\begin{aligned}
\limsup_{M \rightarrow \infty} \frac{1}{M} \left| \int_{B_{M\epsilon}} e^{2\pi i h f(x)} dx \right| &\leq \limsup_{M \rightarrow \infty} \frac{1}{M} \left| \int_{\epsilon M}^M \chi_{B_{M\epsilon}} dx \right| \\
&\leq \limsup_{M \rightarrow \infty} \frac{K M^\beta M^{1-\beta} \sin^{-1} \epsilon + l.o.t.}{M} \quad (4.15) \\
&= K \sin^{-1} \epsilon \\
&= O(\epsilon)
\end{aligned}$$

■

#### 4.4.1 $\alpha \neq k(1 - \beta)$

**Theorem 4.4.3** *For  $\beta < 1$ ,  $\alpha \notin \mathbb{Z}(1 - \beta)$  ( $n^\alpha \sin n^\beta$ ) is uniformly distributed modulo 1.*

In order to prove Theorem 4.4.3 we first define  $f_{m\beta}(x) = \sum_j c_j x^{\alpha_j} \sin(x^\beta + \phi_j)$  where the sum consists of finitely many terms,  $c_j \in \mathbb{R}$  and  $\alpha_j < m(1 - \beta)$ . Let  $k$  be the subscript of the largest  $\alpha_j$  in  $f_{m\beta}(x)$ . Ensure that  $\alpha_k > 0$  and  $\alpha_k \notin \mathbb{Z}(1 - \beta)$  and that the  $\alpha_j$ 's are all different. In addition ensure that  $c_k \neq 0$  and  $-\pi < \phi_j \leq \pi$ .

**Proposition 4.4.4** *( $f_{m\beta}(n)$ ) is uniformly distributed modulo 1.*

Theorem 4.4.3 will then follow directly from Proposition 4.4.4.

#### Proof

We will prove Proposition 4.4.4 by induction on  $m$ . The proof will follow a similar line of argument to that used to prove Proposition 3.3.1 however extra care will need to be taken due to the additional sine term.

Choose  $\beta$  with  $0 < \beta < 1$ . Pick  $\epsilon > 0$ . We now consider the distribution of  $f_{1\beta}$ . Let  $k$  be chosen as above and let  $k'$  be the subscript of the second largest  $\alpha_j$ . Let  $M$  be  $\max(\max_j \{c_j \beta \epsilon\}, \max_j \{c_j \alpha_j\})$  and let  $T$  be the number of terms in the sum of  $f_{1\beta}$ .

1. Pick  $N_1 > \epsilon^{\frac{1-\beta}{\beta}}$  (So that for  $x > \epsilon N_1$ ,  $x^\beta > \epsilon$ ).
2. Pick  $N_2 > \frac{1}{\epsilon} \left( \frac{2\alpha_k}{\beta \epsilon^2} \right)^{\frac{1}{\beta}}$  (So that for  $x > \epsilon N_2$ , and  $\cos(x^\beta + \phi_k) \geq \epsilon$   
 $|\epsilon \beta x^{\alpha_k + \beta - 1} \cos(x^\beta + \phi_k)| > 2|\alpha_k x^{\alpha_k - 1} \sin(x^\beta + \phi_k)|$ )

3. Pick  $N_3 > \frac{1}{\epsilon} \left( \frac{2TM}{c_k \alpha_k} \right)^{\frac{1}{\alpha_k - \alpha_{k'} - \beta}}$  (So that for  $x > \epsilon N_3$ ,

$$\begin{aligned} |c_k \alpha_k x^{\alpha_k - 1} \sin(x^\beta + \phi_k)| &> 2TKx^{\alpha_{k'} - \beta - 1} \\ &> \left| \sum_{j, j \neq k} c_j (\beta x^{\alpha_j + \beta - 1} \cos(x^\beta + \phi_j) + \alpha_j x^{\alpha_j - 1} \sin(x^\beta + \phi_j)) \right| \end{aligned}$$

Pick  $N > \max(N_1, N_2, N_3)$ . We will consider  $\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=\epsilon N}^N e^{2\pi i h f_{1\beta}(n)}$  and show that this goes to zero by bounding the terms of the Euler summation formula. Let the set  $A_{N\epsilon}$  and  $B_{N\epsilon}$  be defined by

$$A_{N\epsilon} = \{x \mid \epsilon N \leq x \leq N, |\cos(x^\beta + \phi_k)| \geq \epsilon\}$$

$$B_{N\epsilon} = [1, N] \setminus A_{N\epsilon}$$

Let us consider the size of the first term in the Euler summation formula. We denote this term by

$$|\text{First Term}| \leq \underbrace{\left| \int_{A_{N\epsilon}} e^{2\pi i h \sum_j c_j x^{\alpha_j} \sin(x^\beta + \phi_j)} dx \right|}_{I_A} + \underbrace{\left| \int_{B_{N\epsilon}} e^{2\pi i h \sum_j c_j x^{\alpha_j} \sin(x^\beta + \phi_j)} dx \right|}_{I_B} \quad (4.16)$$

We begin by simplifying this integral using some of the tricks from section 2.2.

Let

$$A(x) = c_k \beta x^{\alpha_k + \beta - 1} \cos(x^\beta + \phi_k)$$

and

$$B(x) = c_k \alpha_k x^{\alpha_k - 1} \sin(x^\beta + \phi_k) + \sum_{j, j \neq k} c_j (\beta x^{\alpha_j + \beta - 1} \cos(x^\beta + \phi_j) + \alpha_j x^{\alpha_j - 1} \sin(x^\beta + \phi_j))$$

Then

$$\frac{d}{dx} \left( \sum_j c_j x_j^\alpha \sin(x^\beta + \phi_j) \right) = A(x) + B(x)$$

We have:

$$\begin{aligned} \epsilon |A(x)| &= \epsilon c_k \beta |x^{\alpha_k + \beta - 1} \cos(x^\beta + \phi_k)| \\ &> 2c_k \alpha |x^{\alpha_k - 1} \sin(x^\beta + \phi_k)| \\ &\quad (\text{Since } N > N_2) \\ &> c_k \alpha |x^{\alpha_k - 1} \sin(x^\beta + \phi_k)| + \left| \sum_{j, j \neq k} c_j (\beta x^{\alpha_j + \beta - 1} \cos(x^\beta + \phi_j) + \alpha_j x^{\alpha_j - 1} \sin(x^\beta + \phi_j)) \right| \\ &\quad (\text{Since } N > N_3) \\ &\geq |B(x)| \end{aligned} \quad (4.17)$$



so we will be able to apply Proposition 2.2.1. We now use (2.20) to bound  $|I_A|$  from (4.16).

$$\begin{aligned}
|I_A| &= \left| \int_{A_{N\epsilon}} \frac{d}{dx} \left( \frac{e^{2\pi i h \sum_j c_j x_j^\alpha \sin(x^\beta + \phi_j)}}{2\pi i h} \right) \times \right. \\
&\quad \left. \left( \sum_j c_j (\beta x^{\alpha_j + \beta - 1} \cos(x^\beta + \phi_j) + \alpha_j x^{\alpha_j - 1} \sin(x^\beta + \phi_j)) \right)^{-1} dx \right| \\
&\leq K \left| \int_{A_{N\epsilon}} \frac{d}{dx} \left( e^{2\pi i h \sum_j c_j x_j^\alpha \sin(x^\beta + \phi_j)} \right) \frac{dx}{c_k \beta x^{\alpha_k + \beta - 1} \cos(x^\beta + \phi_k)} \right| + \epsilon K' N \\
&\quad \text{(Using 2.21)} \\
&\leq \left[ \frac{K}{|c_k \beta x^{\alpha_k + \beta - 1} \cos(x^\beta + \phi_k)|} \right]_{A_{N\epsilon}} \\
&\quad + K \int_{A_{N\epsilon}} \left| \frac{(\alpha_k + \beta - 1) x^{\alpha_k + \beta - 2} \cos(x^\beta + \phi_k) + \beta x^{\alpha_k + 2\beta - 2} \sin(x^\beta + \phi_k)}{c_k \beta x^{2\alpha_k + 2\beta - 2} \cos^2(x^\beta + \phi_k)} \right| dx + \epsilon K' N \\
&\quad \text{(Using 2.20)} \\
&\leq \left[ \frac{K}{|x^{\alpha_k + \beta - 1} \epsilon|} \right]_{A_{N\epsilon}} + \underbrace{\int_{A_{N\epsilon}} \frac{K_1 \epsilon x^{\alpha_k + \beta - 2} + K_2 x^{\alpha_k + 2\beta - 2}}{x^{2\alpha_k + 2\beta - 2} \epsilon^2} dx}_{J} + \epsilon K' N \\
&\hspace{25em} (4.18)
\end{aligned}$$

Now

$$\begin{aligned}
\left[ \frac{K}{|x^{\alpha_k + \beta - 1} \epsilon|} \right]_{A_{N\epsilon}} &\leq \frac{K N^{1 - \alpha_k - \beta} N^\beta}{\epsilon} \\
&\quad \text{(Using Lemma 4.4.1 with } M = N \text{ and } g(x) = x^\beta + \phi_k) \\
&= \frac{K N^{1 - \alpha_k}}{\epsilon} \\
&\hspace{25em} (4.19)
\end{aligned}$$

We now consider  $|J|$  which was deferred from above.

$$\begin{aligned}
|J| &= \int_{A_{N\epsilon}} \frac{K_1 \epsilon x^{\alpha_k + \beta - 2} + K_2 x^{\alpha_k + 2\beta - 2}}{x^{2\alpha_k + 2\beta - 2} \epsilon^2} dx \\
&\leq \int_{A_{N\epsilon}} \frac{K x^{\alpha_k + 2\beta - 2}}{x^{2\alpha_k + 2\beta - 2} \epsilon^2} dx \\
&\quad \text{(Since } N > N_1\text{)} \\
&\leq \frac{K}{\epsilon^2} \int_{A_{N\epsilon}} x^{-\alpha_k} dx \\
&= \frac{K}{\epsilon^2} [x^{1-\alpha_k}]_{A_{N\epsilon}} \\
&\leq \frac{K}{\epsilon^2} [x^{1-\alpha_k}]_1^N \\
&\leq \frac{K}{\epsilon^2} N^{1-\alpha_k}
\end{aligned} \tag{4.20}$$

Thus combining (4.18), (4.19) and (4.20) gives

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \int_{A_{N\epsilon}} e^{2\pi i h \sum_j c_j x^{\alpha_j} \sin(x^\beta + \phi_j)} dx \right| &\leq \limsup_{N \rightarrow \infty} \left( \frac{KN^{-\alpha_k}}{\epsilon} + \frac{K'N^{-\alpha_k}}{\epsilon^2} + \epsilon K'' \right) \\
&= O(\epsilon)
\end{aligned} \tag{4.21}$$

Now let  $B_{N\epsilon} = [\epsilon N, N] \setminus A_{N\epsilon}$ . Then using Lemma 4.4.2 with  $M = N$  we get

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \int_{B_{N\epsilon}} e^{2\pi i h \sum_j c_j x^{\alpha_j} \sin(x^\beta + \phi_j)} dx \right| = O(\epsilon) \tag{4.22}$$

Substituting (4.21) and (4.22) into (4.16) shows us that  $\limsup_{N \rightarrow \infty} \frac{|\text{First Term}|}{N} = O(\epsilon)$ .

We now look at the third term in the Euler summation formula.

$$\begin{aligned}
|\text{Third term}| &\leq \int_{\epsilon N}^N \sum_j c_j (K_{1j}|x^{\alpha_j+\beta-1} \cos(x^\beta + \phi_k)| + K_{2j}|x^{\alpha_j-1} \sin(x^\beta + \phi_k)|) dx \\
&\leq \sum_j c_j \left( K_{1j} \int_{\epsilon N}^N x^{\alpha_j+\beta-1} dx + K_{2j} \int_{\epsilon N}^N x^{\alpha_j-1} dx \right) \\
&\leq \left[ \sum_j c_j K_{1j} x^{\alpha_j+\beta} + K_{2j} x^{\alpha_j} \right]_{\epsilon N}^N \\
&\leq \sum_j c_j (K_{1j} N^{\alpha_j+\beta} + K_{2j} N^{\alpha_j})
\end{aligned} \tag{4.23}$$

And since  $\alpha_j < (1 - \beta)$  for all  $j$  Then  $(\alpha_j + \beta) < 1$  and dividing by  $N$  and taking the limsup as  $N \rightarrow \infty$  we see that the third term of the Euler summation formula goes to 0 as  $N \rightarrow \infty$ . Making use of (2.17) we have

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{\epsilon N}^N e^{2\pi i h f_{1\beta}(n)} \right| = O(\epsilon)$$

and using Proposition 2.2.5  $(f_{1\beta}(n))$  is u.d. mod 1.

Now let us assume that  $(f_{b\beta}(n))$  is uniformly distributed modulo 1 for all  $b \leq m$  and consider the distribution of  $(f_{(m+1)\beta}(n))$ . Using Taylor expansion gives:

$$\begin{aligned}
f_{m\beta}(n+h) - f_{m\beta}(n) &= h f_{(m+1)\beta}^{(1)}(n) + \frac{h^2}{2} f_{(m+1)\beta}^{(2)}(n) + \cdots + O(n^{-\delta}) \\
&\quad (\text{Where } \delta > 0 \text{ and the constant in the } \\
&\quad O \text{ term depends on } h) \\
&= h f_{m\beta}^*(n) + \frac{h^2}{2} f_{(m-1)\beta}^*(n) + \cdots + \underbrace{O(n^{-\delta})}_{< \epsilon} \\
&= f_{m\beta}^{**}(n) + \epsilon_n
\end{aligned} \tag{4.24}$$

Now the leading coefficient  $f_{m\beta}^{**}(n)$  is  $c_k \beta \neq 0$  and the leading power of  $f_{m\beta}^{**}(n)$  is  $\alpha_k - (1 - \beta) \notin \mathbb{Z}(1 - \beta)$ . Thus  $f_{(\alpha+\beta-1)\beta}^{**}(n)$  satisfies the induction hypothesis and thus is u.d. mod 1. Hence by the difference theorem Proposition 4.4.4 holds.  $\blacksquare$

Thus if  $\alpha$  and  $\beta$  satisfy the hypothesis of Theorem 4.4.3 then  $(n^\alpha \sin n^\beta)$  is uniformly distributed modulo 1.

#### 4.4.2 $\alpha = k(1 - \beta)$

**Theorem 4.4.5** *If  $0 < \beta < 1$  and  $k \in \mathbb{Z}$  then  $(n^{k(1-\beta)} \sin n^\beta)$  is uniformly distributed modulo 1.*

We let  $g_{k\beta}(n) = c_k n^{k(1-\beta)} \sin(n^\beta + \phi) + f_{k\beta}(n)$  where  $f_{k(1-\beta)\beta}(n)$  is defined as above and  $\phi$  is as above. Theorem 4.4.5 will follow as a direct result of the following Proposition.

**Proposition 4.4.6**  *$(g_{k\beta}(n))$  is uniformly distributed modulo 1.*

#### Proof

We will prove Proposition 4.4.6 by induction on  $k$ . We will also need a number of other propositions.

#### Proposition 4.4.7

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i m (g_{1\beta}(n+h) - g_{1\beta}(n))} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i m K h \cos(n^\beta + \phi)}$$

#### Proof

Using Taylor Expansion gives:

$$\begin{aligned} g_{1\beta}(n+h) - g_{1\beta}(n) = & c_1 h \beta \cos(n^\beta + \phi) + \underbrace{c_1 h n^{-\beta} \sin(n^\beta + \phi)}_{< \epsilon} \\ & + \underbrace{c_1 h^2 (n^{\beta-1} \sin(n^\beta + \phi) - \beta n^{-\beta-1} \sin(n^\beta + \phi) + \beta n^{-1} \sin(n^\beta + \phi)) + \dots}_{< \epsilon} \\ & + h \underbrace{\sum_j (c_j \alpha_j n^{\alpha_j-1} \sin(n^\beta + \phi) + c_j \beta n^{\alpha_j-(1-\beta)} \cos(n^\beta + \phi)) + \dots}_{< \epsilon} \end{aligned} \tag{4.25}$$

and so by Corollary 2.2.4

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i m (g_{1\beta}(n+h) - g_{1\beta}(n))} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i m K h \cos(n^\beta + \phi)} + O(\epsilon)$$

and since we can choose  $\epsilon$  to be as small as we want Proposition 4.4.7 follows.

■

**Proposition 4.4.8**

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i m K h \cos(n^\beta + \phi)} \right| \leq \frac{K}{h^{1/3}}$$

**Proof**

Once again we use Euler's summation formula. We let

$$A_{Nh} = \{x | 1 \leq x \leq N, |\cos x^\beta| \geq h^{-1/3}\}$$

$$B_{Nh} = [1, N] \setminus A_{Nh}$$

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{|\text{First Term}|}{N} &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \underbrace{\left| \int_{A_{Nh}} e^{2\pi i m K h \cos(x^\beta + \phi)} dx \right|}_{I_A} \\ &\quad + \limsup_{N \rightarrow \infty} \frac{1}{N} \underbrace{\left| \int_{B_{Nh}} e^{2\pi i m K h \cos(x^\beta + \phi)} dx \right|}_{I_B} \end{aligned} \quad (4.26)$$

Now

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{|I_A|}{N} &= \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \int_{A_{Nh}} \frac{d}{dx} \left( e^{2\pi i m K h \cos x^\beta} \right) (2\pi i m K h \beta x^{\beta-1} \cos x^\beta)^{-1} dx \right| \\ &\leq \limsup_{N \rightarrow \infty} \frac{K}{Nh} \left| \left[ \frac{1}{|x^{\beta-1} \cos(x^\beta + \phi)|} \right]_{A_{Nh}} \right| + \\ &\quad \limsup_{N \rightarrow \infty} \frac{1}{Nh} \int_{A_{Nh}} \frac{|K_1 x^{2\beta-2} \sin(x^\beta + \phi) - K_2 x^{\beta-2} \cos(x^\beta + \phi)|}{|x^{2\beta-2} \cos^2(x^\beta + \phi)|} dx \\ &\quad \text{(Using 2.20)} \\ &\leq \limsup_{N \rightarrow \infty} \frac{K}{Nh} \left[ \frac{x^{1-\beta}}{\cos(x^\beta + \phi)} \right]_{A_{Nh}} + \limsup_{N \rightarrow \infty} \frac{K_1}{Nh} \int_{A_{Nh}} \left| \frac{\sin(x^\beta + \phi)}{\cos^2(x^\beta + \phi)} \right| dx + l.o.t. \, dx \\ &\leq \limsup_{N \rightarrow \infty} \frac{K N^{1-\beta} N^\beta}{Nh \cdot h^{-1/3}} + \limsup_{N \rightarrow \infty} \frac{K_1}{Nh \cdot h^{-2/3}} \int_{A_{Nh}} dx \\ &\quad \text{(Using Lemma 4.4.1 with } M = N \text{ and } g(x) = x^\beta + \phi) \\ &= \limsup_{N \rightarrow \infty} \left( \frac{KN}{Nh^{2/3}} + \frac{K_1 N}{Nh^{1/3}} \right) \\ &\leq Kh^{-1/3} \end{aligned} \quad (4.27)$$

and using Lemma 4.4.2 with  $M = N$  gives

$$\limsup_{N \rightarrow \infty} \frac{|I_B|}{N} = O(h^{-1/3}) \quad (4.28)$$

Finally

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|\text{Third Term}|}{N} &\leq \limsup_{N \rightarrow \infty} \frac{K}{N} \int_1^N |x^{\beta-1} \cos(x^\beta + \phi)| dx \\
&\leq \limsup_{N \rightarrow \infty} \frac{K}{N} \int_1^N x^{\beta-1} dx \\
&\leq \limsup_{N \rightarrow \infty} K N^{\beta-1} \\
&= 0
\end{aligned} \tag{4.29}$$

Combining equations (4.26) - (4.29) proves the proposition.  $\blacksquare$

Using Propositions 4.4.7 and 4.4.8 we see that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{\epsilon N}^N e^{2\pi i h(g_{1\beta}(n+h) - g_{1\beta}(n))} \right| \leq \frac{K}{h^{1/3}}$$

and so we can use the generalised Van Der Corput's Difference Theorem to deduce that  $(g_{1\beta}(n))$  is u.d. mod 1.

Now assume that  $(g_{m\beta}(n))$  is u.d. mod 1 for all  $m < k$ . Using Taylor expansion gives:

$$\begin{aligned}
g_{k\beta}(n+h) - g_{k\beta}(n) &= h g_{k\beta}^{(1)}(n) + \frac{h^2}{2} g_{k\beta}^{(2)}(n) + \dots + O(n^{-\delta}) \\
&\quad (\text{Where } \delta > 0 \text{ and the constant in the } O \text{ term depends on } h) \\
&= h g_{(k-(1-\beta))\beta}^*(n) + \frac{h^2}{2} g_{(k-2(1-\beta))\beta}^*(n) + \dots + \underbrace{O(n^{-\delta})}_{< \epsilon} \\
&= g_{(k-(1-\beta))\beta}^{**}(n) + \epsilon_1
\end{aligned} \tag{4.30}$$

Now the leading coefficient  $g_{(k-(1-\beta))\beta}^{**}(n)$  is  $c_k \beta \neq 0$  and the leading power of  $g_{(k-(1-\beta))\beta}^{**}(n)$  is  $(k-1)(1-\beta)$ . Thus  $g_{k\beta}^{**}(n)$  satisfies the induction hypothesis and so is u.d. mod 1. Hence by the difference theorem Proposition 4.4.6 holds.  $\blacksquare$

**Theorem 4.4.9** *For  $0 < \beta < 1$  and  $0 < \alpha$ ,  $(n^\alpha \sin n^\beta)$  is uniformly distributed modulo 1.*

**Proof**

Combining Theorems 4.4.3 and 4.4.5 proves the theorem.  $\blacksquare$

## 4.5 The distribution of $n^\alpha \sin n$

**Theorem 4.5.1** *If  $\alpha < \frac{1}{2}$  then  $(n^\alpha \sin n)$  is uniformly distributed modulo 1.*

In order to prove the Theorem 4.5.1 we will actually prove the following more general theorem.

**Theorem 4.5.2** *If  $\alpha < \frac{1}{2}$  and  $a \in \mathbb{R}$  then  $n^\alpha \sin 2\pi an$  is uniformly distributed modulo 1.*

### Proof

We will make use of the following theorem due to Dirichlet.

**Theorem 4.5.3** *For any number  $a$  and integer  $M \exists$  a rational  $\frac{p}{q}$  with  $|q| < M$  and  $\left|a - \frac{p}{q}\right| < \frac{1}{qM}$ .*

### Proof

It suffices to find a non-zero integer  $q$  such that  $|q| < M$  and  $||aq|| < \frac{1}{M}$ , where  $||aq||$  is the distance from  $aq$  to the nearest integer.

Consider the numbers  $\{na\}$ ,  $n = 1, \dots, M$ . These are  $M$  numbers in the interval  $[0, 1]$ . By the pigeonhole principle, there must be a pair of them, say  $\{ia\}$  and  $\{ja\}$  with  $|\{ia\} - \{ja\}| < \frac{1}{M}$ . Setting  $q = |i - j|$  proves the theorem. ■

Pick  $\beta$  with  $\alpha < \beta < \frac{1}{2}$ , pick  $\epsilon > 0$  and let  $N_i$ ,  $i = 1, 2$  be chosen as follows

1. Pick  $N_1 > \frac{1}{4\epsilon^4}$
2. Pick  $N_2 > \left(\frac{\alpha}{2\pi\epsilon^3}\right)^{\frac{1}{1-\beta}}$ , so that for  $x > \epsilon(N^{1-\beta}q + r)$  we have  $2\pi\epsilon^2x^\alpha > \alpha x^{\alpha-1}$

Pick  $N > \max(N_1, N_2)$  and pick  $M \sim N^\beta$ .

Using Theorem 4.5.2 we write  $a = \frac{p}{q} + \delta$  noting that  $1 < q < N^\beta$ ,  $q\delta < N^{-\beta}$  and make the change of variables  $n = mq + r$ ,  $m \in \mathbb{Z}$ ,  $0 \leq r < q$ . We then have

$$\sum_{n=\epsilon N}^N e^{2\pi i h n^\alpha \sin 2\pi an} \leq \sum_{r=0}^{q-1} \sum_{m=\lceil \frac{\epsilon N}{q} \rceil}^{\lfloor \frac{N}{q} \rfloor} e^{2\pi i h (mq+r)^\alpha \sin(2\pi a(mq+r))} \quad (4.31)$$

But

$$\begin{aligned}
\sin(2\pi a(mq + r)) &= \sin\left(2\pi\left(\frac{p}{q} + \delta\right)(mq + r)\right) \\
&= \sin\left(2\pi\left(mp + mq\delta + \frac{rp}{q} + r\delta\right)\right) \\
&= \sin\left(2\pi\left(mq\delta + \frac{rp}{q} + r\delta\right)\right) \\
&\quad (\text{Since } mp \in \mathbb{Z})
\end{aligned} \tag{4.32}$$

and substituting (4.32) into (4.31) gives

$$\left| \sum_{n=\epsilon N}^N e^{2\pi i h n^\alpha \sin 2\pi a n} \right| \leq \left| \sum_{r=0}^{q-1} \sum_{m=\lceil \epsilon N/q \rceil}^{\lfloor N/q \rfloor} e^{2\pi i h(mq+r)^\alpha \sin(2\pi(mq\delta + \frac{rp}{q} + r\delta))} \right| + O(q) \tag{4.33}$$

The advantage of this change of variables is that it is now easier for us to apply the Euler summation formula. This is because when differentiating the exponent we now either reduce the power of  $n$  out the front or we pull down a  $q\delta$  which is of size  $\sim N^{-\beta}$ .

We now look at the inner sum of (4.33). Given our choice of  $M$  this sum becomes:

$$\sum_{m=\epsilon q/q}^{N/q} e^{2\pi i h(mq+r)^\alpha \sin(2\pi(mq\delta + \frac{rp}{q} + r\delta))}$$

As we have done previously we will estimate the size of the sum by examining the Euler summation formula. Now let  $g(x) = 2\pi\left(\delta x + \frac{rp}{q}\right)$ ,  $C(x) = 2\pi\delta x^\alpha$  and  $D(x) = \alpha x^{\alpha-1}$ . Define the sets  $A_{N\epsilon r}$  and  $B_{N\epsilon r}$  as follows:

$$A_{N\epsilon r} = \{x \mid \epsilon N \leq x \leq N, |C(x) \cos g(x) + D(x) \sin g(x)| \geq \epsilon \sqrt{C(x)^2 + D(x)^2}\}$$

Using the fact that  $A \sin x + B \cos x = \sqrt{A^2 + B^2} \sin x + \tan^{-1} B/A$  we see that  $A_{N\epsilon r}$  corresponds to the set of  $x$  where

$$\sqrt{C(x)^2 + D(x)^2} |\sin g(x) + \theta(x) + D(x)| \geq \epsilon \sqrt{C(x)^2 + D(x)^2}$$

Where  $\theta(x) = \tan^{-1} \frac{D(x)}{C(x)} = \tan^{-1} \frac{2\pi\delta x}{\alpha}$ . This is the same as the set of  $x$  where  $|\sin g(x) + \theta(x)| \geq \epsilon$ . Thus an equivalent definition of the set  $A_{N\epsilon r}$  is:

$$A_{N\epsilon r} = \{x \mid \epsilon N \leq x \leq N, |\sin g(x) + \theta(x)| \geq \epsilon\}$$



$$B_{N\epsilon r} = [\epsilon N, N] \setminus A_{N\epsilon r}$$

We will soon show the sum over  $r$  of the size of all the  $B_{N\epsilon r}$  sets is small.

The first term of the Euler summation formula is given by:

$$\begin{aligned}
|\text{First Term}| &= \left| \int_{\epsilon N/q}^{N/q} e^{2\pi i h(tq+r)^\alpha \sin(2\pi(tq\delta + \frac{rp}{q} + r\delta))} dt \right| \\
&\leq \left| \frac{1}{q} \int_{\epsilon N}^N e^{2\pi i h x^\alpha \sin(2\pi(\delta x + \frac{rp}{q}))} dx \right| + O(1) \\
&\quad (\text{letting } x = tq + r) \\
&\leq \underbrace{\left| \frac{1}{q} \int_{A_{N\epsilon r}} e^{2\pi i h x^\alpha \sin(2\pi(\delta x + \frac{rp}{q}))} dx \right|}_{I_A} + \underbrace{\left| \frac{1}{q} \int_{B_{N\epsilon r}} e^{2\pi i h x^\alpha \sin(2\pi(\delta x + \frac{rp}{q}))} dx \right|}_{I_B} + O(1)
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
|I_A| &= \frac{K}{q} \left| \int_{A_{N\epsilon r}} \frac{d}{dx} \left( e^{2\pi i h x^\alpha \sin g(x)} \right) \left( \alpha x^{\alpha-1} \sin g(x) + 2\pi \delta x^\alpha \cos g(x) \right)^{-1} dx \right| \\
&= \frac{K}{q} \left| \left[ \frac{1}{\epsilon \sqrt{(\alpha x^{\alpha-1})^2 + (2\pi \delta x^\alpha)^2}} \right]_{A_{N\epsilon r}} \right| \\
&\quad + \frac{K}{q} \underbrace{\left| \int_{A_{N\epsilon r}} e^{2\pi i h x^\alpha \sin g(x)} \frac{d}{dx} \left( \left( \alpha x^{\alpha-1} \sin g(x) + 2\pi \delta x^\alpha \cos g(x) \right)^{-1} \right) dx \right|}_J
\end{aligned} \tag{4.35}$$

We now attempt to invoke as similar argument to Lemma 4.4.1 to evaluate the first term of (4.35). ?? So for each  $x \in A_{N\epsilon r}$  we have

$$\frac{1}{\sqrt{(\alpha x^{\alpha-1})^2 + (2\pi \delta x^\alpha)^2}} < k(N) < N^{-\alpha}$$

We now ask how many intervals there are in the set  $A_{N\epsilon r}$ . We are concerned with the set where  $|\sin g(x) + \theta(x)| \geq \epsilon$ . Let the number of intervals in this set be  $T$ . As in Lemma 4.4.1 we find that if  $|\sin g(x) + \theta(x)| = 0$  then  $g(x) + \theta(x) = t\pi$ ,  $t \in \mathbb{Z}$ . Now  $\theta(x)$  is positive, and is no bigger than  $\frac{\pi}{2}$ , and the  $\frac{rp}{q}$  term of  $g(x)$  can be treated as a constant. Thus we are essentially concerned with how many times  $2\pi\delta x = t\pi$ . In addition  $x < N$ . Thus we solve the inequality

$$\frac{\pi t}{2\pi \delta} < N$$

to find that  $t < 2\delta N$  and can thus conclude that  $T < K\delta N$ . Thus we find that

$$\frac{K}{q} \left| \left[ \frac{1}{\epsilon \sqrt{(\alpha x^{\alpha-1})^2 + (2\pi \delta x^\alpha)^2}} \right]_{A_{N\epsilon r}} \right| \leq \frac{K\delta N N^{-\alpha}}{q\epsilon} \quad (4.36)$$

We now look at the integral  $J$ .

$$\begin{aligned} |J| &\leq \frac{K}{q} \int_{A_{N\epsilon r}} \left| \frac{d}{dx} \left( (\alpha x^{\alpha-1} \sin g(x) + 2\pi \epsilon x^\alpha \cos g(x))^{-1} \right) \right| dx \\ &\leq \frac{K}{q} \int_{A_{N\epsilon r}} \left| \frac{\alpha(\alpha-1)x^{\alpha-2} \sin g(x) + 4\pi \delta \alpha x^{\alpha-1} \cos g(x) + 4\pi^2 \delta^2 x^\alpha \sin g(x)}{(\alpha x^{\alpha-1} \sin g(x) + 2\pi \epsilon x^\alpha \cos g(x))^2} \right| dx \\ &\leq \frac{K}{q} \int_{A_{N\epsilon r}} \frac{K_1 x^{\alpha-2} + K_2 \delta x^{\alpha-1} + K_3 \delta^2 x^{\alpha-2}}{\epsilon^2 (\delta^2 x^{2\alpha} + x^{2\alpha-2})} dx \end{aligned} \quad (4.37)$$

Now by the arithmetic mean, geometric mean inequality we have

$$\delta x^{\alpha-1} \leq \frac{\delta^2 x^\alpha}{2} + \frac{x^{\alpha-2}}{2}.$$

And so we can further bound  $|J|$  by:

$$\begin{aligned} |J| &\leq \frac{K}{q} \int_{A_{N\epsilon r}} \frac{K_1 x^{\alpha-2} + K_2 \delta^2 x^{\alpha-2}}{\epsilon (\delta^2 x^{2\alpha} + x^{2\alpha-2})} \\ &\leq \frac{K}{q} \int_{\epsilon N}^N x^{-\alpha} dx \\ &\leq \frac{K N^{1-\alpha}}{q} \end{aligned} \quad (4.38)$$

Our next aim is to show that  $\sum_{r=1}^q |B_{N\epsilon r}| = O(\epsilon N q)$ . We fix  $x$  and let  $C_{N\epsilon x} = \{r \mid 1 \leq r \leq q, |\sin(2\pi(\delta x + \frac{rp}{q}) + \theta(x))| < \epsilon\}$  ask how big  $C_{N\epsilon r}$  is. Let  $\epsilon_1 = \frac{\sin^{-1} \epsilon}{2\pi}$ . Now if  $\sin(2\pi y + \phi) < \epsilon$  then we must have  $\{y\} \in (K, K + \epsilon_1)$  for some  $K \in [0, 1)$ .

Letting  $y = \frac{rp}{q}$  above we find that the size of  $C_{n\epsilon x}$  is given by  $\phi_q^{\{\frac{kp}{q}\}}(K, K + \epsilon_1)$ . Now  $\{\frac{kp}{q}\}$  takes on  $q$  evenly spaced values in the interval  $[0, 1)$ . Thus there will be approximately  $\epsilon_1 q$  of them in the interval  $[K, K + \epsilon_1)$ . So the size of  $C_{N\epsilon x}$  is  $O(q\epsilon_1) = O(q\epsilon)$ .

Now using Fubini's theorem we can conclude that

$$\begin{aligned}\sum_{r=1}^q |B_{N\epsilon r}| &= \sum_{x=\epsilon N}^N |C_{N\epsilon x}| \\ &= O(\epsilon N q)\end{aligned}\tag{4.39}$$

Thus combining (4.35), (4.37) and (4.39) gives:

$$\begin{aligned}\frac{1}{N} \sum_{r=1}^q |\text{First Term}| &\leq \frac{KqN^{1-\alpha}}{Nq} + \frac{1}{qN} O(\epsilon N q) \\ &\leq KN^{-\alpha} + O(\epsilon)\end{aligned}\tag{4.40}$$

We now consider the third term.

$$\begin{aligned}|\text{Third term}| &= K \int_{\epsilon N/q}^{N/q} \left| q(tq+r)^{\alpha-1} \sin \left( 2\pi \left( tq\delta + \frac{rp}{q} + r\delta \right) \right) \right. \\ &\quad \left. + q\delta(tq+r)^{\alpha} \cos \left( 2\pi \left( tq\delta + \frac{rp}{q} + r\delta \right) \right) \right| dt \\ &\leq K \int_{\epsilon N/q}^{N/q} (q(tq+r)^{\alpha-1} + q\delta(tq+r)^{\alpha}) dt \\ &= K_1 [(tq+r)^{\alpha}]_{\epsilon N/q}^{N/q} + \frac{K_2 q\delta}{q} [(tq+r)^{\alpha+1}]_{\epsilon N/q}^{N/q} \\ &\leq K_1 N^{\alpha} + \frac{K_2 q\delta}{q} N^{\alpha+1} \\ &\quad \text{(Since } r < q < N^{\beta} \text{ the } r \text{ gets absorbed in the constant.)} \\ &\leq \frac{K_1 N}{q} (qN^{\alpha-1}) + \frac{K_2 N}{q} (q\delta N^{\alpha}) \\ &\leq \frac{KN}{q} (N^{(\alpha+\beta)-1} + N^{\alpha-\beta})\end{aligned}\tag{4.41}$$

Note that since  $\alpha < \beta < \frac{1}{2}$  we have  $(\alpha - \beta) < 0$  and  $(\alpha + \beta) - 1 < 0$ . Now

combining (4.33), (4.40) and (4.41) gives

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=\epsilon N}^N e^{2\pi i h n^\alpha \sin 2\pi a n} \right| &\leq \limsup_{N \rightarrow \infty} \frac{1}{q} \sum_{r=0}^q \frac{q}{N} \left| \sum_{m=\lceil \epsilon N/q \rceil}^{\lfloor N/q \rfloor} e^{2\pi i h (mq+r)^\alpha \sin (2\pi (mq\delta + \frac{rP}{q} + r\delta))} \right| \\
&\leq \limsup_{N \rightarrow \infty} \frac{1}{q} \sum_{r=0}^q (K N^{-\alpha} + K' (N^{(\alpha+\beta)-1} + N^{\alpha-\beta}) + O(\epsilon)) \\
&\leq \limsup_{N \rightarrow \infty} K (N^{-\alpha} + N^{(\alpha+\beta)-1} + N^{\alpha-\beta}) + O(\epsilon) \\
&= O(\epsilon)
\end{aligned} \tag{4.42}$$

Thus  $n^\alpha \sin 2\pi a n$  is u.d. mod 1. for  $0 < \alpha < 1$  and  $a$  irrational.  $\blacksquare$

**Proof of Theorem 4.5.1** Letting  $a = \frac{1}{2\pi}$  in Theorem 4.5.2 proves this theorem.  $\blacksquare$

## 4.6 The distribution of $(f(n) \sin n^\beta)$

For a sufficiently slow function  $f$  we find that we can extend the results of Theorem 4.4.3 to include more general families of sequences.

**Theorem 4.6.1** *For  $0 < \beta < 1$ , if  $f(x)$  is a function such that  $f'(x) \leq \frac{K}{x^\gamma}$  with  $\gamma > 1 - \beta$  and  $f(x) \rightarrow \infty$ , then  $(f(n) \sin n^\beta)$  is uniformly distributed modulo 1.*

**Proof**

Pick  $0 < \beta < 1$  and assume  $f(x)$  meets the conditions above. Pick  $0 < \epsilon < 1$  and choose  $N_i$ ,  $i = 1, 2$  as follows:

1.  $N_1$  (So that  $f(\epsilon N_2) > 1$ .)
2.  $N_3 > N_2$  and  $N_3 > \frac{1}{\epsilon} \left(\frac{\beta}{\epsilon^2}\right)^{\frac{1}{\beta+\gamma-1}}$  (So that  $\epsilon \beta x^{\beta-1} f(x) \epsilon > x^{-\gamma} > f'(x) \sin x^\beta$ .)

Then choose  $N > N_2$ .

Once again we use the Euler summation formula and define

$$A_{N\epsilon} = \{x | \epsilon N \leq x \leq N, |\cos x^\beta| \geq \epsilon\}$$

$$B_{N\epsilon} = [\epsilon N, N] / A_{N\epsilon}$$

We look at the size of the first term.

$$|\text{First Term}| \leq \underbrace{\left| \int_{A_{N\epsilon}} e^{2\pi i h f(x) \sin x^\beta} dx \right|}_{I_A} + \underbrace{\left| \int_{B_{N\epsilon}} e^{2\pi i h f(x) \sin x^\beta} dx \right|}_{I_B} \tag{4.43}$$

$$\begin{aligned}
|I_A| &= K \left| \int_{A_{N\epsilon}} \frac{d}{dx} \left( e^{2\pi i h f(x) \sin x^\beta} \right) (f'(x) \sin x^\beta + \beta x^{\beta-1} f(x) \cos x^\beta)^{-1} dx \right| \\
&\leq K \left| \int_{A_{N\epsilon}} \frac{d}{dx} e^{2\pi i h f(x) \sin x^\beta} (2\beta x^{\beta-1} f(x) \cos x^\beta)^{-1} dx \right| + \epsilon K' N \\
&\quad \text{(Since } N > N_3 \text{ we can apply (2.21))} \\
&\leq K \left[ \frac{1}{|x^{\beta-1} f(x) \cos x^\beta|} \right]_{A_{N\epsilon}} + \\
&\quad \int_{A_{N\epsilon}} \left| \frac{K_1 x^{2\beta-2} f(x) \sin x^\beta + K_2 x^{\beta-2} f(x) \cos x^\beta + K_3 x^{\beta-1} f'(x) \cos x^\beta}{x^{2\beta-2} f^2(x) \cos^2 x^\beta} \right| dx + \epsilon K' N \\
&\quad \text{(Using (2.20))} \\
\end{aligned} \tag{4.44}$$

Now observe that since  $f'(x) < Kx^{-\gamma} < Kx^{\beta-1}$  then

$$\begin{aligned}
&|K_1 x^{2\beta-2} f(x) \sin x^\beta + K_2 x^{\beta-2} f(x) \cos x^\beta + K_3 x^{\beta-1} f'(x) \cos x^\beta| \\
&\leq |K_1 x^{2\beta-2} f(x)| + |K_2 x^{\beta-2} f(x)| + |K_3 x^{\beta-1} f'(x)| \\
&\leq K x^{2\beta-2} |f(x)| + K' x^{2\beta-2} \\
&\leq K x^{2\beta-2} |f(x)| \quad \text{(Since } N > N_2 \text{)}
\end{aligned}$$

Thus we have:

$$\begin{aligned}
|I_A| &\leq \frac{KN}{\epsilon f(N)} + K \int_{A_{N\epsilon}} \frac{1}{|f(x) \cos^2 x^\beta|} dx + l.o.t. + \epsilon K' N \\
&\quad \text{(Using Lemma 4.4.1 with } M = N \text{ and } g(x) = x^\beta \text{)} \\
&\leq \frac{KN}{\epsilon f(N)} + \frac{K''}{\epsilon^2} \int_{A_{N\epsilon}} \frac{dx}{f(x)} + l.o.t. + \epsilon K' N \\
&\leq \frac{KN}{\epsilon f(N)} + \frac{K'' N}{\epsilon^2 f(\epsilon N)} + l.o.t. + \epsilon K' N
\end{aligned} \tag{4.45}$$

Once again using Lemma 4.4.2 with  $M = N$  we have

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|I_B|}{N} &= \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \int_{B_{N\epsilon}} e^{2\pi i h f(x) \sin x^\beta} dx \right| \\
&= O(\epsilon)
\end{aligned} \tag{4.46}$$

Combining equations 4.43, 4.45 and 4.46 gives:

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|\text{First Term}|}{N} &\leq \limsup_{N \rightarrow \infty} \left( \frac{K}{\epsilon f(N)} + \frac{K''}{\epsilon^2 f(\epsilon N)} + O(\epsilon) + \underbrace{l.o.t.}_{\rightarrow 0} \right) \\
&= O(\epsilon)
\end{aligned} \tag{4.47}$$

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|\text{Third Term}|}{N} &\leq \limsup_{N \rightarrow \infty} \frac{K}{N} \int_{\epsilon N}^N |x^{\beta-1} f'(x) \cos x^\beta| dx \\
&\leq \lim_{N \rightarrow \infty} \frac{K}{N} \int_{\epsilon N}^N x^{-\gamma+\beta-1} dx \\
&= \lim_{N \rightarrow \infty} \frac{K}{N} [x^{\beta-\gamma}]_{\epsilon N}^N \\
&\leq \lim_{N \rightarrow \infty} \frac{KN^{\beta-\gamma}}{N} \\
&= \lim_{N \rightarrow \infty} KN^{-(\gamma+(1-\beta))} \\
&= 0
\end{aligned} \tag{4.48}$$

Combining equations (4.47) and (4.48) and using Proposition 2.2.5 we see that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N e^{2\pi i h f(x) \sin x^\beta} \right| = O(\epsilon)$$

and so  $(f(x) \sin x^\beta)$  is u.d. mod 1. ■

## Chapter 5

# Applying functions to sequences

In the previous chapters we have analysed the distribution of particular families of sequences. In this chapter we take a different approach and attempt to apply functions to families of sequences about which we already know something. We have already seen an attempt at this in section 4.1 when we discussed the distribution of  $\sin x_n$  where  $(x_n)$  was uniformly distributed modulo  $2\pi$ . Most of the time however we will not be able to arrive at particularly conclusive results about the distribution of such sequences. This is because we can generally find fairly contrived yet valid sequences which serve as a counter example to anything we would like to conclude.

## 5.1 Adding and multiplying uniformly distributed sequences

Let  $(u_n)$  and  $(v_n)$  be sequences which are uniformly distributed modulo 1 and consider the distribution of  $(u_n + v_n)$ . Very little can be said about the above sequence. In some situations it will be u.d. and in others it will not. To illustrate the first of these two cases, let  $u_n = \sqrt{2}n$  and  $v_n = \sqrt{3}n$  then  $u_n + v_n = (\sqrt{2} + \sqrt{3})n$  which is u.d. since  $\sqrt{2} + \sqrt{3}$  is irrational.

To illustrate the second case, let  $u_n$  be as above, and let  $v_n = -\sqrt{2}n$ . Then  $u_n + v_n = 0 \forall n$  and hence is clearly not u.d.

Similarly we consider the distribution of  $(u_nv_n)$  and again arrive at inconclusive results.

When  $u_n = v_n = \sqrt{2}n$  then  $u_nv_n = 2n^2$  which is clearly not u.d. mod 1. However if  $u_n$  is as above and  $v_n = \sqrt{3}n$  then  $u_nv_n = \sqrt{6}n^2$  which is u.d. mod 1 using Proposition 3.1.3.

## 5.2 The distribution of $\log u_n$

We now ask whether the distribution of  $(\log u_n)$  is related to the distribution of  $(u_n)$ .

Certainly if  $(u_n)$  is not u.d. mod 1, then we can have  $(\log u_n)$  not u.d. mod 1. To see this simply let  $u_n = n$ .

Now let  $u_n = n\theta$  where  $\theta$  is irrational. Then  $u_n$  is u.d. mod 1 and

$$\begin{aligned} \sum_{n=1}^N e^{2\pi i \log n \theta} &= \sum_{n=1}^N e^{2\pi i (\log n + \log \theta)} \\ &= e^{2\pi i \log \theta} \sum_{n=1}^N e^{2\pi i \log n} \end{aligned}$$

However dividing by  $N$  and attempting to take the limit as  $N$  approaches infinity we find that the limit on the right hand side does not exist (see proof of Proposition 3.2.1) and thus the limit on the left hand side does not exist and therefore is certainly not zero, hence  $\log n\theta$  is not u.d. mod 1. Thus if  $(u_n)$  is u.d. mod 1, then we can have  $(\log u_n)$  not u.d. mod 1.

Now suppose that  $0 \leq \log u_n < 1$  and that  $(\log u_n)$  is uniformly distributed in the unit interval and so  $(\log u_n)$  is u.d. mod 1. Then  $1 \leq u_n < e$ .



Let  $v_n = \{u_n\}$  and consider the distribution of  $v_n$

$$\begin{aligned}
\lim_{N \rightarrow \infty} \frac{\phi_N^{v_n}(0, e-2)}{N} &= \lim_{N \rightarrow \infty} \frac{\phi_N^{u_n}(1, e-1) + \phi_N^{u_n}(2, e)}{N} \\
&= \lim_{N \rightarrow \infty} \frac{\phi_N^{\log u_n}(0, \log(e-1)) + \phi_N^{\log u_n}(\log 2, \log e)}{N} \\
&= \log(e-1) + 1 - \log 2 \\
&\simeq 0.848 \\
&> e-2
\end{aligned}$$

Hence  $(v_n) = (\{u_n\})$  does not satisfy (1.1) and so is not uniformly distributed in the unit interval. Hence  $(u_n)$  is not uniformly distributed modulo 1.

It is thus possible to have  $(\log u_n)$  u.d. mod 1, and  $(u_n)$  not u.d. mod 1.

Finally we let  $u_n = e^{n\sqrt{2}} - \{e^{n\sqrt{2}}\} + \{n\sqrt{2}\}$ . Then  $\{u_n\} = \{n\sqrt{2}\}$  and so  $u_n$  is uniformly distributed modulo 1. We now consider the distribution of  $(\log u_n)$ . To do this we will use the following fact: Pick  $\epsilon > 0$  and let  $\eta_0 = \frac{1}{\epsilon}$ . If  $\eta_0 < a, b$  and  $|b - a| \leq 1$  using the mean value theorem we have:

$$|\log b - \log a| < \epsilon \quad (5.1)$$

Pick  $N_0$  so that  $e^{N_0\sqrt{2}} > \eta_0 + 1$ . Then since  $|u_n - e^{n\sqrt{2}}| < 1$  we can use (5.1) to show that for  $n > N_0$

$$|\log u_n - \log e^{n\sqrt{2}}| < \epsilon$$

This means that for  $n > N_0$ ,  $\log u_n = n\sqrt{2} + \epsilon_n$  where  $|\epsilon_n| < \epsilon$ . Since  $\epsilon$  can be chosen to be arbitrarily small we can conclude that for  $n > N_0$   $(\log u_n)$  is uniformly distributed modulo 1. This means that  $(\log u_n)$  is uniformly distributed modulo 1 and so it is possible to have both  $(u_n)$  and  $(\log u_n)$  u.d. mod 1.

From the above analysis we can conclude that the distribution of  $(\log u_n)$  is not at all dependent on the distribution of  $(u_n)$ .

### 5.3 The distribution of $u_n + \sin v_n$

**Proposition 5.3.1** *If  $(u_n, v_n)$  is u.d. mod  $(1, 2\pi)$  then  $(u_n + \sin v_n)$  is uniformly distributed modulo 1.*

**Proof**

Let  $y_k = u_k + \sin v_k$  Now  $\{y_k\} \in [0, h)$  iff  $z_k = \{u_k\} + \{\sin v_k\} \in [0, h) \cup [1, 1+h)$ .

Our aim is to show that  $\lim_{N \rightarrow \infty} \frac{\phi_N^{\{y_k\}}(0, h)}{N} = h$  In order to do this we will first consider the joint distribution of  $(\{u_n\}, \{\sin v_n\})$ .

For  $\epsilon > 0$ ,  $a, b \in [0, 1)$  let  $\theta_1 = \sin^{-1} b$ ,  $\gamma_1 = \sin^{-1}(b + \epsilon)$ ,  $\theta_2 = \sin^{-1}(1 - b)$  and  $\gamma_2 = \sin^{-1}(1 - (b + \epsilon))$ . If  $r_k = (\{u_n\}, \{\sin v_n\}) \in (a, a + \epsilon) \times (b, b + \epsilon)$  then

$$\begin{aligned} s_k &= (\{u_n\}, \{v_n\}_{2\pi}) \\ &\in (a, a + \epsilon) \times [(\theta_1, \gamma_1) \cup (\pi - \gamma_1, \pi - \theta_1) \cup \\ &\quad (\pi + \gamma_2, \pi + \theta_2) \cup (2\pi - \theta_2, 2\pi - \gamma_2)] \end{aligned}$$

And so we have

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{N} \phi_N^{r_k}((a, a + \epsilon) \times (b, b + \epsilon)) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} (\phi_N^{s_k}((a, a + \epsilon) \times (\theta_1, \gamma_1)) + \phi_N^{s_k}((a, a + \epsilon) \times (\pi - \gamma_1, \pi - \theta_1)) \\ &\quad + \phi_N^{s_k}((a, a + \epsilon) \times (\pi + \gamma_2, \pi + \theta_2)) + \phi_N^{s_k}((a, a + \epsilon) \times (2\pi - \theta_2, 2\pi - \gamma_2))) \\ &= (a + \epsilon - a) \frac{(\gamma_1 - \theta_1) + (\pi - \theta_1 - \pi + \gamma_1) + (\pi + \theta_2 - \pi - \gamma_2) + (2\pi - \gamma_2 - 2\pi + \theta_2)}{2\pi} \\ &\quad \text{(Since } (u_n, v_n) \text{ is u.d. mod } (1, 2\pi)) \\ &= \frac{2\epsilon}{2\pi} ((\sin^{-1}(b + \epsilon) - \sin^{-1} b) + (\sin^{-1}(1 - (b + \epsilon)) - \sin^{-1}(1 - b))) \\ &= \frac{\epsilon^2}{\pi} \left( \frac{1}{\sqrt{1 - b^2}} + \frac{1}{\sqrt{1 - (1 - b)^2}} \right) + O(\epsilon^3) \\ &\quad \text{(by Taylor expansion)} \end{aligned} \tag{5.2}$$

We now consider  $\phi_N^{\{y_k\}}(0, h)$  for  $N$  large.

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \phi_N^{\{y_k\}}(0, h) &= \lim_{N \rightarrow \infty} \frac{1}{N} (\phi_N^{z_k}(0, h) + \phi_N^{z_k}(1, 1 + h)) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \left( \lim_{\epsilon \rightarrow 0} \sum_{\beta=0}^{\frac{1}{\epsilon}-1} \sum_{\alpha=\frac{1}{\epsilon}-\beta-1}^{\frac{1}{\epsilon}-1} \phi_N^{r_k}(E_{\alpha\beta\epsilon}) \right) \tag{5.3} \\ &\quad \text{where } E_{\alpha\beta\epsilon} = (\epsilon\alpha, \epsilon(\alpha + 1)) \times (\epsilon\beta, \epsilon(\beta + 1)) \end{aligned}$$

Now when ever  $r_k$  is in  $\sum_{\beta=-1}^{\frac{1}{\epsilon}} \sum_{\alpha=\frac{1}{\epsilon}-\beta-2}^{\frac{1+h}{\epsilon}-\beta} E_{\alpha\beta\epsilon}$  Then  $z_k$  is in  $(0, h) \cup (1, 1 + h)$  and whenever  $z_k$  is in  $(0, h) \cup (1, 1 + h)$  then  $r_k$  is in  $\sum_{\beta=2}^{\frac{1}{\epsilon}-2} \sum_{\alpha=\frac{1}{\epsilon}-\beta}^{\frac{1+h}{\epsilon}-\beta-2} E_{\alpha\beta\epsilon}$

and so we can evaluate  $\lim_{N \rightarrow \infty} \frac{1}{N} \phi_N^{\{y_k\}}(0, h)$  by evaluating the integral associated with the right hand side of (5.3), making use of (5.2). We get:

$$\begin{aligned}
\lim_{N \rightarrow \infty} \frac{1}{N} \phi_N^{\{y_k\}}(0, h) &= \frac{1}{\pi} \int_0^1 \int_{1-y}^{1+h-y} \left( \frac{1}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-(1-y)^2}} \right) dx dy \\
&= \frac{h}{\pi} \int_0^1 \left( \frac{1}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-(1-y)^2}} \right) dy \\
&= \frac{h}{\pi} [\sin^{-1} y - \sin^{-1} (1-y)]_0^1 \\
&= \frac{h}{\pi} \left[ \frac{\pi}{2} - 0 - 0 + \frac{\pi}{2} \right] \\
&= h
\end{aligned} \tag{5.4}$$

and so  $(y_k)$  is u.d. mod 1. ■

## 5.4 The distribution of $\{u_n\} \sin v_n$

**Proposition 5.4.1** *If  $(u_n)$  is uniformly distributed modulo 1 and  $(v_n)$  is uniformly distributed modulo  $2\pi$  then  $(\{u_n\} \sin v_n)$  is not uniformly distributed modulo 1.*

### Proof

Let  $S_{1N} = \{n : n < N, \{u_n\} < \epsilon\}$  and  $S_{2N} = \{n : n < N, 0 < \sin v_n < \epsilon\}$ . Now assume that  $(\{u_n\} \sin v_n)$  is u.d. mod 1, then

$$\begin{aligned}
\epsilon &= \lim_{N \rightarrow \infty} \frac{\phi_N^{\{u_n\} \sin v_n}(0, \epsilon)}{N} \\
&> \lim_{N \rightarrow \infty} \frac{|S_{1N} \cup S_{2N}|}{N} \\
&= \lim_{N \rightarrow \infty} \frac{|S_{1N}| + |S_{2N}| - |S_{1N} \cap S_{2N}|}{N} \\
&> \epsilon + f(\epsilon) - \epsilon^2
\end{aligned} \tag{5.5}$$

Where  $f(\epsilon) = \lim_{N \rightarrow \infty} \frac{\phi_N^{\sin v_n}(0, \epsilon)}{N}$ . And thus from (5.5) we get

$$f(\epsilon) < \epsilon^2 \tag{5.6}$$

Now let  $\gamma = \sin^{-1} \epsilon$  then

$$\begin{aligned}
f(\epsilon) &= \lim_{N \rightarrow \infty} \frac{\phi_N^{\sin v_n}(0, \epsilon)}{N} \\
&= \lim_{N \rightarrow \infty} \frac{\phi_N^{v_n}(0, \gamma) + \phi_N^{v_n}(\pi - \gamma, \pi)}{N} \\
&= \frac{(\gamma - 0) + (\pi - \pi + \gamma)}{2\pi} \tag{5.7} \\
&\quad \text{(Since } (v_n) \text{ is u.d. mod } 2\pi.) \\
&= \frac{\gamma}{\pi}
\end{aligned}$$

Taking  $\epsilon = 0.1$  we get  $f(\epsilon) = 0.032 > \epsilon^2 = 0.01$  which contradicts (5.6).  
Thus  $(\{u_n\} \sin v_n)$  cannot be u.d. mod 1. ■

# Appendix A

## The distribution of $n \sin n$

In this appendix we attempt simplify a proof given in [2] to show the uniform distribution of the sequence  $(n \sin n)$ . The proof of [2] actually falls down in one particular case. While an attempt was made to understand this case, lack of time meant that we could not write it up in time for this report.

**Theorem A.0.2** *For  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  the sequence  $(n \sin 2\pi\alpha n)$  is uniformly distributed modulo 1.*

In the course of the proof we will make use of the following Lemma which is known as Van Der Corput's estimate.

**Lemma A.0.3** *Let  $g \in C^2[X_1, X_2]$  and assume that  $0 < \lambda \leq g''(x) \leq \frac{\lambda}{\epsilon}$  then*

$$\left| \sum_{x=X_1}^{X_2} e^{2\pi i g(x)} \right| \leq \frac{1}{\epsilon} \left( X \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} \right) \quad (\text{A.1})$$

where  $X = X_2 - X_1$ .

### Proof

(See [9])

If  $\lambda > 1$  then we just estimate the left-hand side trivially by  $X$ . So we may assume that  $\lambda \leq 1$ .

We shall shortly show the estimate

$$\sum_{n=A}^B e^{2\pi i g(n)} = 2\lambda^{-1/2} \quad (\text{A.2})$$

whenever there exists an integer  $M$  such that

$$M - \frac{1}{2} \leq g'(n) \leq M + \frac{1}{2}$$

for all  $A \leq n \leq B$ .

Assume (A.2) for the moment. From the hypotheses on  $g''$  we see that  $g'$  is increasing and that  $g'(X_2) - g'(X_1) \leq \frac{X\lambda}{\epsilon}$ . Thus we can partition the left-hand side of (A.1) into  $\frac{X\lambda+1}{\epsilon}$  summations of the form (A.2), so that we can estimate the left-hand side of (A.1) by

$$\frac{(X\lambda + 1)\lambda^{-1/2}}{\epsilon^2} \leq \frac{1}{\epsilon^2}(X\lambda^{1/2} + \lambda^{-1/2})$$

as desired.

Now we prove (A.2). We may set  $M = 0$  by the trick of by replacing  $g(n)$  by  $g(n) - nM$  (note that this does not affect (A.2)). Thus we have

$$-\frac{1}{2} \leq g'(n) \leq \frac{1}{2}$$

for all  $A \leq n \leq B$ .

By dividing the sum into three smaller sums if necessary, we can assume that one of the following three statements is true:

Case 1:  $-1/2 \leq g'(n) \leq -\lambda^{1/2}$  for all  $A \leq n \leq B$ .

Case 2:  $-\lambda^{1/2} < g'(n) < \lambda^{1/2}$  for all  $A \leq n \leq B$ .

Case 3:  $\lambda^{1/2} \leq g'(n) \leq 1/2$  for all  $A \leq n \leq B$ .

We will not consider Case 1 as it is very similar to Case 3.

Consider Case 2. By the hypothesis on  $g''$  there are at most  $2\lambda^{-1/2}$  terms in the sum, so we may trivially bound this sum by  $2\lambda^{-1/2}$ , which is OK.

Now consider Case 3. We shall use a summation by parts trick.

From the MVT we have

$$g(n+1) = g(n) + g'(n + \theta_n)$$

for some  $0 \leq \theta_n \leq 1$ . Thus

$$e^{2\pi i g(n)} = e^{2\pi i g(n+1)} - e^{2\pi i g(n)} \frac{1}{e^{2\pi i g'(n+\theta_n)} - 1}.$$

Since

$$\frac{1}{e^{i\theta} - 1} = \frac{1}{2} \frac{e^{i\theta} + 1}{e^{i\theta} - 1} - \frac{1}{2} = \frac{i}{2} \cot(\theta/2) - \frac{1}{2}$$

we have

$$\begin{aligned} \sum_{n=A}^B e^{2\pi i g(n)} &= \sum_{n=A}^{B-1} e^{2\pi i g(n)} + O(1) \\ &= \sum_{n=A}^{B-1} \left( e^{2\pi i g(n+1)} - e^{2\pi i g(n)} \left( \frac{i}{2} \cot(\pi g'(n + \theta_n)) - \frac{1}{2} \right) \right) + O(1). \end{aligned}$$

By the summation by parts formula previously seen as (3.19) we can rewrite this as

$$e^{2\pi ig(B)} \left( \frac{i}{2} \cot(\pi g'(B-1+\theta_{B-1})) - \frac{1}{2} \right) - e^{2\pi ig(A)} \left( \frac{i}{2} \cot(\pi g'(A+\theta_A)) - \frac{1}{2} \right) \\ - \sum_{n=A+1}^{B-1} e^{2\pi ig(n)} \left( \frac{i}{2} \cot(\pi g'(n+\theta_n)) - \frac{i}{2} \cot(\pi g'(n-1+\theta_{n-1})) \right) + O(1)$$

The first two terms are no bigger than  $\lambda^{-1/2}$  since we are in Case 3 and  $\cot(\theta) = \frac{1}{\theta}$  for  $0 < \theta < \frac{\pi}{2}$ . Now look at the third term. The absolute value of this term is less than

$$\sum_{n=A+1}^{B-1} |\cot(\pi g'(n+\theta_n)) - \cot(\pi g'(n-1+\theta_{n-1}))|.$$

But in Case 3, the sequence  $\cot(\pi g'(n+\theta_n))$  is decreasing in  $n$ , so we can telescope this series as

$$\cot(\pi g'(A+1+\theta_{A+1})) - \cot(\pi g'(B-1+\theta_{B-1}))$$

which is no bigger than  $\lambda^{-1/2}$  since we are in Case 3. ■

Lemma A.0.3 can actually be generalised to work for all derivatives of  $g$ . It's general form is stated below:

**Lemma A.0.4** *Let  $g \in C^2[X_1, X_2]$  and assume that  $0 < \lambda_j \leq g^{(j)}(x) \leq K\lambda_j$  then letting  $J = 2^j$  we have*

$$\left| \sum_{x=X_1}^{X_2} e^{2\pi ig(x)} \right| \leq K \left( X\lambda^{1/(2-J)} + 1 + X^{1-2/J} + X(\lambda X^{4-8/J})^{-2/J} \right) \quad (\text{A.3})$$

where  $X = X_2 - X_1$ .

We will use this Lemma for the case when  $j = 3$  without proof.

**Proof of Theorem A.0.2**

As usual we pick  $\epsilon > 0$ . Let  $\epsilon_2 = \sin^{-1} \epsilon$  and pick  $b \in \mathbb{Z}$  with  $b \neq 0$ . We now choose  $N$  to be “large enough” depending on  $b$  and  $\epsilon$ . Let  $M \sim N^{\frac{5}{8}}$  and invoke Theorem 4.5.3 to write  $\alpha = \frac{p}{q} + \frac{1}{Q}$  with  $q < N^{\frac{5}{8}}$  and  $\frac{q}{Q} < N^{\frac{-5}{8}}$ . Without loss of generality let us assume that  $Q, q > 0$ .

Throughout the proof we let

$$S = \left| \sum_{n=\frac{N}{2}}^N e^{2\pi i b n \sin 2\pi \alpha n} \right| \quad (\text{A.4})$$

Our aim will be to show that  $S < KN\epsilon$ . This will mean that dividing by  $N$  and taking the limsup as  $N \rightarrow \infty$  will give  $\limsup_{N \rightarrow \infty} \frac{S}{N} = 0$  and so making use of Proposition 2.2.5 we can conclude that  $(n \sin 2\pi\alpha n)$  is uniformly distributed modulo 1.

The proof now breaks into two cases, the first where  $Q \geq \frac{N}{\epsilon_2}$  and so  $\alpha$  is close to a rational, and the second where  $Q \leq \frac{N}{\epsilon_2}$  and so  $\alpha$  is far from being a rational.

Case 1:  $Q > \frac{N}{\epsilon_2}$

We now write  $n = mq + k$  with  $0 \leq k \leq q$ . Then

$$\sin 2\pi\alpha n = \sin \left( 2\pi \left( \frac{p}{q} + \frac{1}{Q} \right) (mq + k) \right) = \sin \left( 2\pi \left( \frac{p}{q} + \frac{1}{Q} \right) (mq + k) \right)$$

For fixed  $k$  let  $f(m) = 2\pi \left( \frac{kp}{q} + \frac{mq+k}{Q} \right)$  and  $g(m) = b(mq + k) \sin f(m)$  and note that

$$g''(m) = K \frac{q^2}{Q} \cos f(m) + K' \left( \frac{q}{Q} \right)^2 (mq + k) \sin f(m)$$

Let us now denote by  $B_k$  the set of  $k$ 's for which either  $\sin f(m) < \epsilon$  or  $\cos f(m) < \epsilon$  and so one of the terms of  $g''$  is very small. These are  $k$ 's for which either

$$0 < \left\{ \frac{f(m)}{2\pi} \right\} < \sin^{-1} \epsilon \quad \text{or} \quad \cos^{-1} \epsilon < \left\{ \frac{f(m)}{2\pi} \right\} < 1$$

Now  $\frac{mq+k}{Q} < \frac{N}{Q} < \epsilon_2 = \sin^{-1} \epsilon$  and so if  $0 < \left\{ \frac{kp}{q} \right\} < 2\sin^{-1} \epsilon$  then sine condition is satisfied, and a similar argument works for the cosine condition. Hence for  $q$  large enough the size of  $B_k$  can be approximated by  $\phi_q^{\left\{ \frac{2\pi kp}{q} \right\}}(0, 2\epsilon_2) + \phi_q^{\left\{ \frac{2\pi kp}{q} \right\}}(1 - 2\epsilon_2, 1)$ . Now  $\left\{ \frac{kp}{q} \right\}$  takes on  $q$  evenly spaced values in the interval  $[0, 1)$ . Thus for large enough  $q$  there will be approximately  $2\epsilon_2 q$  of them in the interval  $[0, 2\epsilon_2)$  and the interval  $[1 - 2\epsilon_2, 1)$ . Thus the size of  $B_k$  is  $O(q\epsilon)$ . Let  $A_k$  denote the set of all other  $k$ 's. Then we have

$$\begin{aligned} S &\leq \sum_{A_k} \left| \sum_{m=1}^{\frac{N}{q}} e^{2\pi i g(m)} \right| + \sum_{B_k} \left| \sum_{m=1}^{\frac{N}{q}} e^{2\pi i g(m)} \right| \\ &\leq \sum_{A_k} \left| \sum_{m=1}^{\frac{N}{q}} e^{2\pi i g(m)} \right| + Kq\epsilon \cdot \frac{N}{q} \\ &\leq \sum_{A_k} \left| \sum_{m=1}^{\frac{N}{q}} e^{2\pi i g(m)} \right| + KN\epsilon \end{aligned} \tag{A.5}$$



Now for  $k \in A_k$  we have:

$$\begin{aligned}
|g''(m)| &= \left| K \frac{q^2}{Q} \cos f(m) + K' \left( \frac{q}{Q} \right)^2 (mq + k) \sin f(m) \right| \\
&\leq K \frac{q^2}{Q} + K \epsilon_2 \frac{q^2}{Q} \\
&\leq K \frac{q^2}{Q}
\end{aligned} \tag{A.6}$$

In addition,

$$\begin{aligned}
|g''(m)| &= \left| K \frac{q^2}{Q} \cos f(m) + K' \left( \frac{q}{Q} \right)^2 (mq + k) \sin f(m) \right| \\
&\geq K \frac{q^2}{Q} \epsilon_2 - K \left( \frac{q}{Q} \right)^2 (mq + k) \epsilon_2 \\
&\geq K \frac{q^2}{Q} \epsilon_2 - K \frac{q^2}{Q} \epsilon_2^2 \\
&\geq K \frac{q^2}{Q} \epsilon_2
\end{aligned} \tag{A.7}$$

This case now splits into three subcases.

Case 1a:  $\frac{N^{5/4}}{\epsilon^2 \epsilon_2^2} < Q < \epsilon^2 \epsilon_2^2 N^2$

In this case we can apply Lemma A.0.3 with  $\lambda = \frac{q^2}{Q} \epsilon_2$ . Note that:  $\left( \frac{q^2}{\epsilon^2} \right) < \frac{N^{5/4}}{\epsilon^2} < Q$  so  $\frac{q^2}{Q} < \epsilon^2$ . We get:

$$\begin{aligned}
\left| \sum_m e^{2\pi i g(m)} \right| &\leq \frac{1}{\epsilon_2} \left( \frac{N}{q} \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} \right) \\
&= \frac{N}{q \epsilon_2} \left( \left( \frac{q^2}{Q} \right)^{1/2} + \frac{q \sqrt{Q}}{N q} \right) \\
&\leq \frac{N}{q \epsilon_2} (\epsilon \epsilon_2 + \epsilon \epsilon_2) \\
&\leq \frac{2N \epsilon}{q}
\end{aligned} \tag{A.8}$$

Case 1b:  $Q < \frac{N^{5/4}}{\epsilon^2 \epsilon_2^2}$

In this case we are required to look at the third derivative of  $g$  and use the

general form of Van Der Corput's estimate. We have:

$$g^{(3)}(m) = K \frac{q^3}{Q^2} \sin f(m) + K' \left( \frac{q}{Q} \right)^3 \cos f(m)$$

As previously we can bound  $g^{(3)}$  from above and below via:

$$\begin{aligned} |g^{(3)}(m)| &= \left| K \frac{q^3}{Q^2} \sin f(m) + K' \left( \frac{q}{Q} \right)^2 (mq + k) \cos f(m) \right| \\ &\leq K \frac{q^3}{Q^2} + K \epsilon_2 \frac{q^2}{Q} \\ &\leq K \frac{q^3}{Q^2} \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} |g^{(3)}(m)| &= \left| K \frac{q^3}{Q^2} \sin f(m) + K' \left( \frac{q}{Q} \right)^3 (mq + k) \cos f(m) \right| \\ &\geq K \frac{q^3}{Q^2} \epsilon_2 - K \left( \frac{q}{Q} \right)^3 (mq + k) \epsilon_2 \\ &\geq K \frac{q^3}{Q^2} \epsilon_2 - K \frac{q^3}{Q^2} \epsilon_2^2 \\ &\geq K \frac{q^3}{Q^2} \epsilon_2 \end{aligned} \tag{A.10}$$

We can apply Lemma A.0.4 with  $\lambda_3 = \frac{q^3}{Q^2}$ . Observe that  $\frac{q^3}{Q^2} < \frac{N^{15/8}}{N^2} = N^{-1/8}$ . We get:

$$\begin{aligned} \left| \sum_m e^{2\pi i g(m)} \right| &\leq K \left( \frac{N}{q} \left( \frac{q^3}{Q^2} \right)^{1/6} + 1 + \left( \frac{N}{q} \right)^{3/4} + \frac{N}{q} \left( \frac{Q^2}{q^3} \left( \frac{N}{q} \right)^3 \right)^{-1/4} \right) \\ &= \frac{KN}{q} \left( N^{-1/48} + \frac{q}{N} + \left( \frac{q}{N} \right)^{3/4} + \left( \frac{Q^2}{N^3} \right)^{-1/4} \right) \\ &\leq \frac{KN}{q} \left( N^{-1/48} + N^{-3/8} + N^{-9/32} + \left( \frac{N^{-3/4}}{\epsilon^2 \epsilon_2^2} \right)^{1/4} \right) \\ &\leq \frac{KN N^{-1/48}}{q} \quad (\text{For } N \text{ large enough}) \\ &\leq \frac{KN \epsilon}{q} \quad (\text{For } N \text{ large enough}) \end{aligned} \tag{A.11}$$

We now join Cases 1a and 1b together and show that we can put an appropriate bound on  $S$ .

Using (A.4) and (A.8) we get

$$\begin{aligned}
S &\leq \sum_{A_k} \left| \sum_m e^{2\pi i g(m)} \right| + \sum_{B_k} \left| \sum_m e^{2\pi i g(m)} \right| \\
&\leq \sum_{A_{nk}} \frac{H\epsilon}{q} + KN\epsilon \\
&\leq q \cdot \frac{N\epsilon}{q} + KN\epsilon \\
&= KN\epsilon
\end{aligned} \tag{A.12}$$

This is the appropriate bound for  $S$  and so if  $\frac{N}{\epsilon} < Q, \epsilon N^2$  then the sequence is u.d. mod 1.

Case 1c:  $Q > \epsilon N$

The proof of this case in [2] actually falls down for this case when considering the sequence  $(n \sin 2\pi \alpha n)$ . It is possible to find a proof that  $S = O(N\epsilon)$  in this case, however limited time meant that we could not write it up.

Case 2:  $Q \leq \frac{N}{\epsilon_2}$

We begin by setting  $H = \epsilon^3 \epsilon_2^5 Q$ . We now break the sum from (A.4) up into smaller but overlapping sums of length  $H$  and sum these up. The result is that each term from (A.4) gets summed  $H$  times, and in addition a few extra terms at the end are added. We find that

$$\begin{aligned}
S &\leq \frac{1}{H} \left| \sum_{n=\frac{N}{2}-H}^N \sum_{h=H}^{2H} e^{2\pi i b(n+h) \sin 2\pi \alpha(n+h)} - \underbrace{\text{extra terms which are added at the end}}_{< H} \right| \\
&\leq \frac{1}{H} \sum_{n=\frac{N}{2}-3H}^N \left| \sum_{h=H}^{2H} e^{2\pi i b(n+h) \sin 2\pi \alpha(n+h)} \right| + N\epsilon \\
&\quad (\text{Since } H < \epsilon^3 \epsilon_2^5 Q < c\epsilon^3 \epsilon_2^4 N < N\epsilon)
\end{aligned} \tag{A.13}$$

We now write  $h = mq + k$  and observe that:

$$\sin 2\pi \alpha(n+h) = \sin \left( 2\pi \alpha n + 2\pi \left( \frac{p}{q} + \frac{1}{Q} \right) (mq+k) \right) = \sin \left( 2\pi \alpha n + 2\pi \left( \frac{kp}{q} + \frac{h}{Q} \right) \right).$$

For fixed  $n, k$  let  $f(m) = 2\pi \alpha n + 2\pi \left( \frac{kp}{q} + \frac{mq+k}{Q} \right)$  and let

$g(m) = b(n + mq + k) \sin f(m)$ . We see that

$$g''(m) = K \left( \frac{q}{Q} \right)^2 (n + h) \sin f(m) + K' \frac{q^2}{Q} \cos f(m) \quad (\text{A.14})$$

In addition we can now write  $S$  as:

$$S \leq \frac{1}{H} \sum_{n,k} \left| \sum_m e^{2\pi i g(m)} \right| + N\epsilon \quad (\text{A.15})$$

Where  $n$  ranges from 1 to  $N$ ,  $k$  ranges from 1 to  $q$  and  $m$  ranges from  $\frac{H}{q}$  to  $\frac{2H}{q}$ . We would like to examine the size of the inner sum of (A.15). In order to do this we will attempt to find a  $\lambda$  such that  $\lambda < |g''| < K\lambda$  and then invoke Lemma A.0.3. Unfortunately there are problem values of  $(n, k)$  for which  $g''(m)$  is very small or even vanishes. In order to avoid this we will break the set of  $(n, k)$  pairs up into the “good” set and the “problem” set and show that the “problem” set is not particularly large.

There are now two sub cases. Either  $Q < KN\epsilon$  and so the dominant term in the second derivative is the first one, or  $Q \geq KN\epsilon$  in which case the two terms of the second derivative are of roughly the same size. We will consider these two cases separately.

Case 2a:  $Q < KN\epsilon$

Let us denote by  $B_{nk}$  the set of pairs  $(n, k)$  which have  $\sin f(m) < \epsilon_2$  and by  $A_{nk}$  the remaining pairs. Now  $\frac{mq+k}{Q} < \frac{H}{Q} = \epsilon^3 \epsilon_2^5$  and so this term of  $f(m)$  makes very little difference to  $\sin f(m)$ . Thus the size of  $B_{nk}$  is essentially the same as  $\sum_{k=1}^q \phi_N^{\{n\alpha + \frac{kp}{q}\}}(0, \epsilon)$ . And since  $(n\alpha)$  is uniformly distributed modulo 1, for  $N$  large enough this is about  $qN\epsilon$ . Thus we get

$$\begin{aligned} \frac{1}{H} \sum_{B_{nk}} \left| \sum_m e^{2\pi i g(m)} \right| &\leq \frac{1}{H} \cdot qN\epsilon \cdot \frac{H}{q} \\ &= N\epsilon \end{aligned} \quad (\text{A.16})$$

Now over the set  $A_{nk}$  we have  $\epsilon_2 \leq \sin f(m) \leq 1$  and so since the first term of  $g''$  is the dominant one and since  $(n + h) > \frac{N}{2} - 3H = O(N)$  we have:  $K \left( \frac{q}{Q} \right)^2 N\epsilon_2 < |g''(m)| < K \left( \frac{q}{Q} \right)^2 N$  and we can apply Lemma A.0.3 with  $\lambda = \left( \frac{q}{Q} \right)^2 N$ .

Case 2b:  $Q \geq KN\epsilon$

In this case we rewrite the  $g''$  as follows.

$$g''(m) = K \left( \frac{q}{Q} \right)^2 \sqrt{(n+h)^2 + K'Q^2} \sin(f(m) + 2\pi\theta(n)) \quad (\text{A.17})$$

Where  $\theta(n) = \frac{1}{2\pi} \tan^{-1} \left( \frac{KQ}{n+h} \right)$ .

Let us denote by  $B_{nk}$  the set of pairs  $(n, k)$  which have  $\sin(f(m) + 2\pi\theta(n)) < \epsilon_2$  and by  $A_{nk}$  the remaining pairs. As before the  $\frac{mq+k}{Q}$  term of  $f(m)$  makes very little difference to  $\sin f(m) + \theta(n)$ . Thus the size of  $B_{nk}$  is essentially the same as  $\phi_N^{\{n^\alpha + \frac{kp}{q} + \theta(n)\}}(0, \epsilon)$ . Now  $\theta(n)$  is a decreasing function ranging over the domain  $[1, n]$ . We can break this domain up into subintervals  $I_i = [n_{i-1}, n_i]$ ,  $i = 1 \dots \frac{1}{4\epsilon}$  with  $1 = n_0 < n_1 < \dots < n_{\frac{1}{4\epsilon}}$  such that for  $n \in I_i$  we have  $\theta(n) = \theta(n_i) + O(\epsilon)$ . Now for  $|I_i| > M_0$  where  $M_0$  depends only on epsilon, we certainly have

$$\begin{aligned} \phi_{[n_{i-1}, n_i]}^{\{n^\alpha + \frac{kp}{q} + \theta(n)\}}(0, \epsilon) &= \phi_{[n_{i-1}, n_i]}^{\{n^\alpha + \frac{kp}{q} + \theta(n_i) + O(\epsilon)\}}(0, \epsilon) \\ &< K\epsilon(n_i - n_{i-1}) \end{aligned} \quad (\text{A.18})$$

and for  $|I_i| < M_0$  then we at least have

$$\phi_{[n_{i-1}, n_i]}^{\{n^\alpha + \frac{kp}{q} + \theta(n)\}}(0, \epsilon) < M_0 \quad (\text{A.19})$$

Combining (A.18) and (A.19) we get

$$\begin{aligned} \phi_N^{\{n^\alpha + \frac{kp}{q} + \theta(n)\}}(0, \epsilon) &= \sum_{i=1}^{\frac{1}{4\epsilon}} \phi_{[n_{i-1}, n_i]}^{\{n^\alpha + \frac{kp}{q} + \theta(n)\}}(0, \epsilon) \\ &\leq \sum_{i=1}^{\frac{1}{4\epsilon}} (K\epsilon(n_i - n_{i-1}) + M_0) \\ &= KN\epsilon + \frac{K'M_0}{\epsilon} \\ &\leq KN\epsilon \\ &\quad (\text{For } N \text{ large enough since } M_0 \text{ does not depend on } N) \end{aligned} \quad (\text{A.20})$$

Thus the size of  $B_{nk}$  is essentially the same as it was above and we find that (A.16) holds in this case as well.

Now over the set  $A_{nk}$  we have  $\epsilon_2 \leq \sin f(m) \leq 1$ . In addition we observe that  $KN\epsilon < Q < \sqrt{(n+h)^2 + K'Q^2} < N$  and so

$$K \left( \frac{q}{Q} \right)^2 N\epsilon\epsilon_2 < |g''(m)| < K \left( \frac{q}{Q} \right)^2 N$$

and as in the first sub case we can apply Lemma A.0.3 with  $\lambda = \left(\frac{q}{Q}\right)^2 N$ .

We now join both sub cases back together and apply Lemma A.0.3 over the good  $k$ 's. Observe that

$$\lambda < N^{-\frac{10}{8}} N = N^{-\frac{1}{4}}$$

and

$$\frac{q}{H} = \frac{q}{\epsilon^3 \epsilon_2^5 Q} < \frac{N^{-\frac{5}{8}}}{\epsilon^3 \epsilon_2^5} = O(N^{-\frac{1}{2}})$$

We get

$$\begin{aligned} \left| \sum_m e^{2\pi i g(m)} \right| &\leq K \left( \frac{H}{q} \lambda^{\frac{1}{2}} + 1 + \lambda^{-\frac{1}{2}} \right) \\ &= \frac{KH}{q} \left( \lambda^{\frac{1}{2}} + \left( \frac{q}{H} \right) + \frac{q}{H \lambda^{\frac{1}{2}}} \right) \\ &\leq \frac{KH}{q} \left( N^{-\frac{1}{4}} + N^{-\frac{1}{2}} + \frac{q}{\epsilon^3 \epsilon_2^5 Q} \frac{Q}{q \sqrt{N}} \right) \\ &\leq \frac{KH N^{-\frac{1}{4}}}{q} \\ &\quad \text{(For } N \text{ large enough)} \\ &\leq \frac{KH \epsilon}{q} \\ &\quad \text{(For } N \text{ large enough)} \end{aligned} \tag{A.21}$$

And using (A.15), (A.16) and (A.21) we get

$$\begin{aligned} S &\leq \frac{1}{H} \sum_{A_{nk}} \left| \sum_m e^{2\pi i g(m)} \right| + \frac{1}{H} \sum_{B_{nk}} \left| \sum_m e^{2\pi i g(m)} \right| + KN\epsilon \\ &\leq \frac{1}{H} \sum_{A_{nk}} \frac{H\epsilon}{q} + KN\epsilon \\ &\leq \frac{1}{H} \cdot Nq \cdot \frac{H\epsilon}{q} + KN\epsilon \\ &= KN\epsilon \end{aligned} \tag{A.22}$$

This is the bound we were after, and so we can conclude that for  $Q < \frac{N}{\epsilon_2}$  then the sequence is uniformly distributed modulo 1.  $\blacksquare$

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