Some connections between Oscillatory Integrals

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The purpose of this talk is to:

- Briefly survey the known results concerning the Bochner-Riesz family of conjectures
- Show that the Bochner-Riesz conjecture implies the restriction conjecture
- Discuss the analogous situation for parabolic Bochner-Riesz and restriction
- Show that the Bochner-Riesz conjecture is equivalent to the weak-type endpoint Bochner-Riesz conjecture
- Show that the Kakeya and Nikodym conjectures are equivalent
- Discuss analogues of these results for the local smoothing conjecture

Let $n \geq 2$. The (spherical) Bochner-Riesz conjecture states that the Fourier multipliers

$$\widehat{S^\delta f}(\xi) = (1 - |\xi|^2)_+^\delta \hat{f}(\xi)$$

are bounded on L^p whenever $\delta > n|1/p - 1/2| - 1/2, 0$.

- These operators have applications to the convergence of spherically summed Fourier series, and discrete versions of these multipliers appear in number theory.
- The multiplier is singular on the sphere and so cannot be treated by classical multiplier theorems.
- Apart from the trivial case $p = 2$, $\delta = 0$, the conditions on δ are necessary. (Herz, Fefferman)
- It suffices to consider $1 \leq p \leq 2n/(n+1)$. The critical case is $2n/(n+1)$.
- The case $n = 2$ is very well studied, and most questions concerning Bochner-Riesz multipliers and related operators have been completely settled. (Carleson, Sjölin, Carbery, Cordoba, Seeger, ...). However these techniques do not seem to extend to general n.

Summary of progress on the conjecture, for n=3

- True for $p' = \infty$. (Bochner)
- True for $p' \geq 6$. (Fefferman, Stein, 1970)
- True for $p' \geq 4$. (Stein, Tomas, 1975; refined and extended by Strichartz, Greenleaf, Christ, Sogge, T., ...)
- True for $p' \geq 4 - \frac{8}{75}$. (Bourgain, 1991)
- True for $p' \geq 4 - \frac{2}{15}$. (Bourgain, 1995)
- True for $p' \geq 4 - \frac{2}{11}$. (Wolff, 1995)
- Critical case is $p' = 3$.

Denote by $BR(p, \alpha)$ the Hörmander-type estimate

$$\| \int e^{-2\pi i R|x-y|} a(x,y) f(y)\ dy\|_p \lesssim R^{-\frac{n}{p'}+\alpha} \|f\|_p$$

where the bump function a is supported away from $x = y$. The Bochner-Riesz conjecture is equivalent to $BR(p, \alpha)$ holding for all $1 \leq p \leq 2n/(n+1)$ and $\alpha > 0$. (Carleson, Sjölin)

$BR(p, \alpha)$ fails when $\alpha < 0$. (We will address the endpoint estimate $BR(p, 0)$ later).

The (spherical) Restriction conjecture states that whenever $f \in L^p(\mathbf{R}^n)$ for $1 \le p < 2n/(n+1)$, then $\hat{f}$ has a well-defined restriction to the sphere:

$$\|\mathfrak{R}f\|_p = \|\hat{f}|_{S^{n-1}}\|_p \lesssim \|f\|_{L^p(\mathbf{R}^n)}.$$

- The conjecture is known to fail for the critical exponent $p = 2n/(n+1)$.
- The exponent p on the left-hand side is negotiable. If the conjecture was true, one could use interpolation to replace the left-hand exponent by q for any $1 \le q \le \frac{n-1}{n+1}p'$.
- Progress on the restriction conjecture has paralleled the progress on the Bochner-Riesz conjecture. For example, the $n = 2$ case is completely solved. A restriction estimate with $q = 2$ is known to imply the corresponding Bochner-Riesz estimate (Fefferman), but such estimates are not available near the critical case.

Summary of progress on the conjecture, for n=3

- True for $p' = \infty$. (Riemann-Lebesgue)
- True for $p' \geq 6$. (Stein, 1967)
- True for $p' > 4$. (Tomas, 1975)
- True for $p' \geq 4$. (Stein, 1975)
- True for $p' > 4 - \frac{2}{15}$. (Bourgain, 1991)
- True for $p' > 4 - \frac{2}{11}$. (Wolff, 1995; sharpened by Moyua-Vargas-Vega, 1996)
- Critical case is $p' = 3$.

Denote by $R(p, \alpha)$ the Hörmander-type estimate

$$\| \int e^{-2\pi i R \langle \frac{x}{|x|}, y \rangle} a(x, y) f(y)\ dy \|_p \lesssim R^{-\frac{n}{p'} + \alpha} \|f\|_p,$$

where the bump function a is supported away from $x = 0$. $R(p, 0)$ is equivalent to the restriction conjecture for the exponent p, while $R(p, \alpha)$ is equivalent to the localized restriction theorem

$$\|\hat{f}|_{S^{n-1}}\|_p \lesssim R^\alpha \|f\|_{L^p(B(0,R))}$$

whenever f is supported on $B(0, R)$. Localized restriction conjectures were used by Bourgain as stepping stones to the full restriction conjecture.

Connection between Bochner-Riesz and Restriction

The oscillatory integral versions of $BR(p,\alpha)$ and $R(p,\alpha)$ are clearly related:

$$\| \int e^{-2\pi i R|x-y|} a(x,y) f(y)\ dy \|_p \lesssim R^{-\frac{n}{p'}+\alpha} \|f\|_p$$

$$\| \int e^{-2\pi i R\langle \frac{x}{|x|}, y\rangle} a(x,y) f(y)\ dy \|_p \lesssim R^{-\frac{n}{p'}+\alpha} \|f\|_p.$$

The phase function for $R(p,\alpha)$ is essentially the linearization of that of $BR(p,\alpha)$:

$$R|x-y| = R|x| - R\langle \frac{x}{|x|}, y\rangle + O(1) \quad \text{for } |x| \sim 1, |y| \lesssim R^{-1/2}.$$

From this we can obtain the relationship

$$BR(p,\alpha) \Rightarrow R(p, 2\alpha).$$

Unfortunately, the global restriction conjecture requires $\alpha = 0$, and so this is not a complete proof that Bochner-Riesz implies restriction. To finish the proof we need the following "epsilon-removal" result:

$$[R(p_0,\epsilon) \quad \forall \epsilon > 0] \Rightarrow [R(p,0) \quad \forall 1 \leq p < p_0].$$

Combining the two implications we see that the full Bochner-Riesz conjecture implies the full restriction conjecture.

We'll outline the proof in the next few transparencies.

Step I: Phase linearization: $BR(p,\alpha) \Rightarrow R(p,2\alpha)$

- If we apply the above approximation to $BR(p,\alpha)$, rescale and eliminate trivial factors, we obtain

  $\| \int e^{-2\pi i (R\langle \frac{x}{|x|}, y\rangle + O(1))} a(x,y) f(y)\, dy\|_p \lesssim R^{-\frac{n}{p'}+2\alpha} \|f\|_p,$

  which would be $R(p,2\alpha)$ but for the $O(1)$ phase error.

- To finish this step, we use the principle that oscillatory integral estimates are stable under smooth $O(1)$ perturbations of the phase or amplitude. (Proof: First consider multiplication by $\exp(2\pi i(k \cdot x + l \cdot y))$, then use Fourier series to handle the general case). [This observation has been made by several authors, including Christ, Bourgain, T., ...]

- One can recast this argument in terms of the original Bochner-Riesz and restriction operators. The key observation is that

  $$S^{\delta(p)+\alpha} f(x) \sim \frac{e^{\pm 2\pi i |x|}}{|x|^{\frac{n}{p}+\alpha}} \mathfrak{R} f(\pm \frac{x}{|x|})$$

  when x is far from the support of f.

Step II: Reduction to sparse sets

- Assume $R(p_0, \varepsilon)$ holds for all $\varepsilon > 0$. We wish to show that $R(p, 0)$ holds for all $1 \leq p < p_0$.
- By interpolation theory it suffices to check the following estimate for characteristic functions:

$$\|\mathfrak{R}\chi_E\|_{p_0} \lesssim |E|^{\frac{1}{p_0}+\varepsilon}.$$

  By the uncertainty principle we may assume E is the union of $1/100$-cubes. (cf. the stability lemma)

- If E were supported in a ball of radius R, then we could employ the hypothesis $R(p_0, \varepsilon)$ and obtain

$$\|\mathfrak{R}\chi_E\|_{p_0} \lesssim |E|^{\frac{1}{p_0}} R^{\varepsilon}.$$

  So if E had a diameter of $O(|E|^N)$ for some N, we would be done.

- So, the bad case occurs when E is thinly spread over a huge ball. For example, E could be contained in an R^C-seperated set of R-balls. The following decomposition lemma shows that this is essentially the only case.

- **Definition.** A collection of R-balls with cardinality M is said to be *sparse* if they are $R^C M^C$-seperated. (C is a fixed large constant).

- **Lemma.** If E is the union of $1/100$-cubes and $N \gg 1$, then there exists $O(N|E|^{1/N})$ sparse collections of balls (with radii $O(|E|^{C^N})$) which cover E.

- This lemma is proven by the usual Calderón-Zygmund type stopping-time arguments. For each $x \in \mathbf{R}^n$, let $k(x) \in \mathbf{Z}^+$ be the first number such that

$$|E \cap B(x, |E|^{C^k})| \leq |E|^{k/N}.$$

The sets $E_k = \{x \in E : k(x) = k\}$ can be covered by $|E|^{1/N}$ sparse collections of $|E|^{C^{k-1}}$ balls, with the cardinality of each collection at most $|E|$.

- In order to use the lemma to finish the proof, we need to bootstrap the estimate

$$\|\mathfrak{R}\chi_E\|_{p_0} \lesssim |E|^{\frac{1}{p_0}} R^\varepsilon$$

to the case when E is not supported on a single R-ball, but rather on a sparse collection of R-balls. This will be done in the next transparency.

Step III: Bootstrapping $R(p_0, \alpha)$ to sparse sets

- $R(p_0, \alpha)$ states that

$$\|\mathfrak{R} f\|_{p_0} \lesssim R^\alpha \|f\|_{p_0}$$

whenever f is supported on a ball of radius R. To finish the proof we need to bootstrap this to

$$\|\sum_i \mathfrak{R} f_i\|_{p_0} \lesssim R^\alpha \|\sum_i f_i\|_{p_0}$$

whenever the f_i are supported on different R-balls in a sparse collection.

- The idea is to exploit the fact that $\hat{f}_i$ and $\hat{f}_j$ behave "independently" on the sphere. More precisely, we have the quasi-orthogonality estimate

$$\|\sum_i \mathfrak{R} f_i\|_p \lesssim (\sum_i \|\tilde{\mathfrak{R}} f_i\|_p^p)^{1/p}$$

for all $1 \leq p \leq 2$, where $\tilde{\mathfrak{R}}$ is a technical perturbation of $\mathfrak{R}$. $p = 1$ is just the triangle inequality, and $p = 2$ will follow from almost orthogonality and sparseness. This finishes the proof.

Remarks

- The epsilon-removal argument actually shows that
$$R(p, \alpha) \Rightarrow R(p - \frac{C}{\log(1/\alpha)}, 0).$$
Thus, this argument only gives a meaningful restriction theorem for exponentially small α. (This may be compared with the implication
$$R(p, \alpha) \Rightarrow R(2 + \frac{1}{\frac{n+1}{2p'} - \alpha}, 0)$$
(Bourgain), which is used in all the recent progress on the restriction conjecture.)

- The argument can be generalized to arbitrary oscillatory integrals (at least, if the amplitudes are smooth and compact). The phases $|x - y|$ and $\langle \frac{x}{|x|}, y \rangle$ can be replaced by more general functions $\phi(x, y)$ and $y \cdot \nabla_y \phi(x, 0)$.

The morals of the story

- The first moral is that any optimal or near-optimal estimate can be scaled into an optimal or near-optimal linear (or polynomial) approximation of that estimate. The idea is to restrict the operator near a critical counterexample, perform the linear (or polynomial) approximation, and rescale.

  Examples:

spherical Bochner-Riesz	$\Rightarrow$	spherical restriction
parabolic Bochner-Riesz	$\Rightarrow$	parabolic restriction
spherical restriction	$\Rightarrow$	parabolic restriction
Nikodym maximal	$\Rightarrow$	Kakeya maximal
Local smoothing	$\Rightarrow$	conic hyperplane Radon

- The second moral is that any epsilons in a near-optimal estimate can usually be removed by "interpolating" with a trivial estimate (e.g. $R(1,0)$).

  Examples:

$$\begin{array}{rcl} R(p_0,\varepsilon) & \Rightarrow & R(p,0) \\ BR(p_0,\varepsilon) & \Rightarrow & BR(p,0) \\ K(p_0,\varepsilon) & \Rightarrow & K(p,0) \\ N(p_0,\varepsilon) & \Rightarrow & N(p,0) \end{array}$$

Parabolic Bochner-Riesz and restriction

- Analogues of the spherical Bochner-Riesz and restriction conjectures exist for paraboloids. As Hörmander problems they can be written as

  $$\| \int e^{-2\pi i R|\underline{x}-\underline{y}|^2/(x_n-y_n)} a(x,y) f(y)\, dy\|_p \lesssim R^{-\frac{n}{p'}+\alpha} \|f\|_p$$

  $$\| \int e^{-2\pi i R(\langle \underline{x},\underline{y}\rangle + |\underline{x}|^2 y_n)} a(x,y) f(y)\ dy\|_p \lesssim R^{-\frac{n}{p'}} \|f\|_p$$

  respectively.

- These problems are closely related to Strichartz estimates for the Schrödinger equation.

- The linearization and epsilon-removal argument extends easily to the parabolic situation. Thus the parabolic Bochner-Riesz conjecture implies the parabolic restriction conjecture.

- Because of the algebraic properties of the two phases, it turns out that one can reverse the implication (Carbery). The main ideas are a projective change of variables $x_n \to 1/x_n$ and an unfreezing argument.

- It also turns out that there is an implication between spherical restriction and parabolic restriction conjecture. This is easiest to see in the context of $L^p \to L^q$ estimates on the sharp line $q = (n-1)p'/(n+1)$.

- The idea is to write spherical restriction as

  $$\|\int e^{-2\pi i R(\langle \underline{x}, \underline{y}\rangle + \phi(\underline{x}) y_n)} a(x,y) f(y)\, dy\|_q \lesssim R^{-\frac{n}{p'}} \|f\|_p.$$

  If $q = (n-1)p'/(n+1)$ one can rescale as

  $$\|\int e^{-2\pi i R(\langle \underline{x}, \underline{y}\rangle + \lambda^2 \phi(\underline{x}/\lambda) y_n)} a(x, y/\lambda) f(y)\, dy\|_q \lesssim R^{-\frac{n}{p'}} \|f\|_p,$$

  and if one lets $\lambda \to \infty$ one obtains the parabolic restriction conjecture. (cf. the osculation arguments of Wolff). One can also obtain a result for (p,q) near the sharp line using the (parabolic) linearization and epsilon removal arguments.

- At the endpoint $p = q = 2n/(n+1)$ the estimate is on the sharp line. Thus the full spherical restriction conjecture implies the full parabolic restriction conjecture.

Weak-type endpoint Bochner-Riesz

- Let $1 \leq p < 2n/(n+1)$. The Bochner-Riesz conjecture for p states that $BR(p, \alpha)$ holds for all $\alpha > 0$, and is equivalent to the L^p boundedness of the multiplier operators S^δ whenever
$$\delta > \delta(p) = \frac{n}{p} - \frac{n+1}{2}.$$
- This conjecture is known to be sharp: S^δ cannot be bounded on L^p for $\delta \leq \delta(p)$ (Herz). However, it is possible that $S^{\delta(p)}$ is of weak-type (p,p). This is the weak-type endpoint Bochner-Riesz conjecture.
- Heuristically speaking, the endpoint conjecture is related to the endpoint estimate $BR(p, 0)$. Thus one can hope to use epsilon-removal techniques to prove the weak-type conjecture.
- **Theorem.** If the Bochner-Riesz conjecture holds for some $1 < p_0 \leq 2n/(n+1)$, then the weak-type endpoint Bochner-Riesz conjecture holds for all $1 \leq p < p_0$.

Summary of progress on the conjecture, for n=3

- True for $p' = \infty$. (Christ, 1988)
- True for $p' > 4$. (Christ, 1987)
- Full conjecture true for $n = 2$. (Seeger, 1996)
- True for $p' \geq 4$. (T., 1996)
- True for $p' > 4 - \frac{2}{11}$. (T., 1997)
- The critical case is $p' = 3$.

The weak-type endpoint Bochner-Riesz conjecture implies the standard Bochner-Riesz conjecture by interpolation. Thus the two conjectures are actually equivalent.

Analogues of this result exist for more general Riesz means (cf. Christ, Sogge, T.)

Sketch of proof

- We use a standard dyadic decomposition

$$S^{\delta(p)} f = \sum_{j>0} 2^{-jn/p} K_j * f + \text{error terms},$$

where $K_j(x) \sim e^{\pm 2\pi i|x-y|}$ for $|x-y| \sim 2^j$.

- By real interpolation techniques and the Calderón-Zygmund decomposition (Seeger), it suffices to show the strong-type estimate

$$\| \sum_J 2^{-jn/p} K_j * f_J \|_p \lesssim (\sum_J \|f_J\|_p^p)^{1/p}$$

where f_J are any functions supported on J, 2^j is the sidelength of J, and J ranges over all dyadic cubes with $j > 0$. By the uncertainty principle we may assume the f_J are constant on 1/100-cubes.

- The Bochner-Riesz hypothesis states that $BR(p_0, \varepsilon)$ holds for some $p_0 > p$. This gives us an estimate

$$\|K_j * f_J\|_p \lesssim 2^{\varepsilon j} \|f_J\|_{p_0}.$$

If f_J is dispersed over a set of size at least $2^{C\varepsilon j}$, then the L^{p_0} norm is smaller than the L^p norm of f, and one can obtain the strong type estimate relatively easily.

- If f_J is concentrated in a ball of radius $2^{C\varepsilon j}$, then $K_j * f_J$ looks like the restriction operator applied to f_J, and are almost orthogonal as J ranges over all dyadic cubes. So one can use quasi-orthogonality techniques to obtain the estimate.

- The bad case is when f_J is concentrated on a set of measure $2^{C\varepsilon j}$ but is very thinly spread over J. Then one uses the sparse Calderón-Zygmund decomposition to reduce matters essentially to the case when each f_J is supported on a sparse collection of relatively small balls. One then uses quasi-orthogonality techniques to obtain the estimate.

Kakeya and Nikodym conjectures

- A Kakeya set (or Besicovitch set) is a set in $\mathbf{R}^n$ that contains a unit line segment in each direction in S^{n-1}. A Nikodym set is a set such that every point in $\mathbf{R}^n$ is incident to a line that contains a unit line segment in the set. It is conjectured that both types of sets must have full dimension n.

- Let $K(p, \alpha)$ denote the estimate

$$\| \sup_{T // \frac{x}{|x|}} \int_T a(x,y) f(y)\ dy \|_p \lesssim R^{-\frac{n}{p'}+\alpha} \|f\|_p,$$

  where T ranges over all $1 \times 1/R$ tubes parallel to $x/|x|$. The Kakeya maximal conjecture states that $K(p, \alpha)$ for all $1 \leq p \leq n$ and $\alpha > 0$. $K(p, \varepsilon)$ implies that Kakeya sets have dimension at least p.

- The Nikodym maximal conjecture is similar but involves the estimate $N(p, \alpha)$:

$$\| \sup_{x \in T} \int_T a(x,y) f(y)\ dy \|_p \lesssim R^{-\frac{n}{p'}+\alpha} \|f\|_p.$$

Summary of progress on the conjectures, for n=3

- True for $p = 1$. (Trivial)
- True for $p \leq 2$. (Drury, 1985)
- True for $p \leq 7/3$. (Bourgain, 1991) (Schlag, 1997)
- True for $p \leq 5/2$. (Wolff, 1995)
- The critical case is $p = 3$.

The Nikodym and Kakeya conjectures are the non-oscillatory counterparts of the (spherical/parabolic) Bochner-Riesz and restriction conjectures. The restriction conjectures are known to imply the Kakeya conjecture (Fefferman, Cordoba). In the reverse direction, it is possible to use progress on Kakeya and Nikodym to obtain a small amount of progress on the Bochner-Riesz and restriction conjectures. (Bourgain)

The situation is more complicated in curved space. (Sogge, Minicozzi)

It turns out that the Kakeya and Nikodym conjectures are equivalent, at least on the maximal function level:

$$K(p, \alpha) \Rightarrow N(p, \alpha) \Rightarrow K(p, 2\alpha).$$

Sketch of equivalence

- If $x \sim 1$ then an $1 \times 1/R$ tube containing x is essentially equivalent to a tube parallel to $x/|x|$, when one restricts oneself to the region $|y| \lesssim R^{-1/2}$. (cf. the connection between Bochner-Riesz and restriction). If one applies this observation and rescales one obtains $N(p, \alpha) \Rightarrow K(p, 2\alpha)$.

- A Nikodym set has a line segment collinear with every point in $\mathbf{R}^n$. Projectively, a Kakeya set has a line segment collinear with every point on the hyperplane at infinity. Thus, by a projective transformation $x_n \to 1/x_n$ and an unfreezing argument, every Nikodym set can be converted into a Kakeya set (cf. the argument by Carbery). Thus the Kakeya set conjecture implies the Nikodym set conjecture.

- The maximal operator analogue of this argument yields the implication $K(p, \alpha) \Rightarrow N(p, \alpha)$.

Local smoothing for the wave equation

- The local smoothing conjecture states that, if u is the solution to the homogeneous wave equation with initial data $u(0) = f$, $u_t(0) = 0$, then

$$\|u\chi_{[1,2]}(t)\|_{L^{2n/(n-1)}_{t,x}} \lesssim \|\nabla^{\varepsilon} f\|_{2n/(n-1)}.$$

- Hölder's inequality and Plancherel's theorem can be used to show that the local smoothing conjecture implies the square-function version of the Bochner-Riesz conjecture (Sogge). By Sobolev imbedding this implies the maximal Bochner-Riesz conjecture for $p > 2$, which implies the Bochner-Riesz conjecture (Stein, Weiss). (The maximal Bochner-Riesz conjecture is partially false for $p < 2$).

- If $F(x,t)$ is a function supported on a fixed compact set in R^{n+1}, we define the conic hyperplane Radon transform $RF(\omega, s)$ for $\omega \in S^{n-1}$, $s \in R$ by

$$RF(\omega, s) = \int F(x, s + \langle \omega, x \rangle)\, dx.$$

  If one linearizes the (adjoint) local smoothing conjecture one obtains the following estimate on R:

$$\|\nabla^{\frac{n-1}{2} - \varepsilon} RF\|_{2n/(n+1)} \lesssim \|F\|_{L^{2n/(n+1)}_{x,t}}.$$

- Similarly, if one linearizes the square function Bochner-Riesz conjecture, one obtains

$$\|\nabla^{\frac{n-1}{2}-\varepsilon} RF\|_{2n/(n+1)} \lesssim \|F\|_{L_x^{2n/(n+1)} L_t^2}.$$

- Either of these estimates implies the spherical restriction conjecture. The idea is to apply the estimate to a function of the form

$$F(x,t) = f(x/\lambda) e^{2\pi i \lambda t} a(x,t)$$

to obtain a localized restriction conjecture, and then use epsilon-removal techniques.

Summary of known equivalences

$$
\begin{array}{ccc}
\text{Local smoothing} & \Rightarrow & \text{conic Radon} \\
\Downarrow & & \Downarrow \\
\text{Square fn. spherical BR} & \Rightarrow & L^2 \text{ conic Radon} \\
\Downarrow & & \Downarrow \\
\text{Maximal spherical BR} & & \Downarrow \\
\Downarrow & & \Downarrow \\
\text{Weak-type spherical BR} & & \text{Spherical} \\
\text{Spherical BR} & \Rightarrow & \text{restriction} \\
 & & \Downarrow \\
\text{Weak-type parabolic BR} & & \text{Parabolic} \\
\text{Parabolic BR} & \Longleftrightarrow & \text{restriction} \\
 & & \Downarrow \\
\text{Nikodym maximal} & \Longleftrightarrow & \text{Kakeya maximal} \\
\Downarrow & & \Downarrow \\
\text{Nikodym set} & \Leftarrow & \text{Kakeya set}
\end{array}
$$