THE REAL INTERPOLATION OF LORENTZ SPACES

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ABSTRACT. We discuss the real interpolation of Lorentz spaces.

1. Lorentz spaces

Let X be a measure space. For any integer k, we define $F_k(X)$ to be the space of all functions f on X which are supported on a set of measure at most 2^k and which satisfy $\|f\|_\infty \leq 1$.

Definition 1.1. For any $0 < p \leq \infty$ and $1 \leq q \leq \infty$, we define the Lorentz space $L^{p,q}(X)$ to be the space of all functions f of the form

$$f = \sum_k c_k 2^{-k/p} f_k$$

where $f_k \in F_k(X)$ and non-negative $c_k \in l^q$, with the $L^{p,q}$ norm being the infimum of the set of all possible $\|c\|_q$.

For a proof that this definition corresponds to the usual definition of $L^{p,q}$, see [1]. The space $L^{p,p}$ is strong L^p, the space $L^{p,\infty}$ is weak L^p, and the space $L^{p,1}$ has a unit ball which is the convex hull of the normalized characteristic functions $|E|^{-1/p}\chi_E$, if $p \geq 1$. Note that $L^{p,q_1} \subset L^{p,q_2}$ when $q_1 \leq q_2$.

Exercise: if $1 < p, q < \infty$ then the spaces $L^{p,q}$ and $L^{p',q'}$ are dual.

2. From restricted weak-type to strong type

Let T be a linear operator from functions on X to functions on Y; let us make the a priori assumption that T is bounded from every L^p to every L^q, but with possibly a large constant; our bounds below will not depend on this constant.

We say that T is of strong type (p, q) if T maps L^p to L^q, and is of restricted weak-type (p, q) if T maps $L^{p,1}$ to $L^{q,\infty}$. Suppose $1 < p, q < \infty$. By duality, the former statement is equivalent to

$$|\Lambda(f, g)| \lesssim \|f\|_p \|g\|_{q'}$$

and the latter is equivalent to

$$|\Lambda(f, g)| \lesssim \|f\|_{p,1} \|g\|_{q',1} \tag{1}$$

where Λ is the bilinear form

$$\Lambda(f, g) = \langle Tf, g\rangle.$$

Since $L^{p,1}$ and $L^{q',1}$ are atomic spaces, (1) is equivalent to the statement that

$$|\Lambda(f_k, g_l)| \lesssim 2^{k/p} 2^{l/q'} \tag{2}$$

for all integers k, l and all $f_k \in F_k(X)$, $g_l = F_l(Y)$.

We now show how restricted weak-type estimates can be bootstrapped to strong type estimates.

Proposition 2.1 (Special case of Marcinkiewicz interpolation)**.** *Let* $1 < p_0 < \infty$*, and suppose* T *is of restricted weak-type* (p, p) *for all* p *in a neighbourhood of* p_0*. Then* T *is of strong type* (p_0, p_0)*.*

Proof We have to show that

$$|\Lambda(f, g)| \lesssim \|f\|_{p_0} \|g\|_{p_0'}.$$

By our characterization of L^{p_0}, it suffices to show that

$$|\Lambda(\sum_k c_k 2^{-k/p_0} f_k, \sum_l d_l 2^{-l/p_0'} g_l\rangle| \lesssim 1$$

where $f_k \in F_k(X)$, $g_l = F_l(Y)$, $\|c\|_{p_0} \lesssim 1$ and $\|d\|_{p_0'} \lesssim 1$.

By linearity we may estimate the left-hand side by

$$\sum_k \sum_l c_k d_l 2^{-k/p_0} 2^{-l/p_0'} |\Lambda(f_k, g_l)|,$$

which by (2) and our hypothesis is estimated by

$$\sum_k \sum_l c_k d_l 2^{-k/p_0} 2^{-l/p_0'} \inf_p 2^{k/p} 2^{l/p'}.$$

Optimizing this in p, we estimate this by

$$\sum_k \sum_l c_k d_l 2^{-\varepsilon|k-l|}.$$

We re-arrange this as

$$\sum_s 2^{-\varepsilon|s|} \sum_k c_k d_{k+s}.$$

Using Hölder and our normalization on c, d, this is bounded by

$$\sum_s 2^{-\varepsilon|s|} \lesssim 1$$

as desired. ■

Interpolating from one restricted weak-type estimate to another is even easier: if T is of restricted weak-type (p_i, q_i) for $i = 0, 1$ and $1 < p_i, q_i < \infty$ then it is also of restricted weak-type (p_θ, q_θ) for all $0 \leq \theta \leq 1$ where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and similarly for q_θ; this follows simply by taking various geometric means of (2) applied to $(p, q) = (p_0, q_0)$ and $(p, q) = (p_1, q_1)$.

3. Interpolation below L^1.

When $q \leq 1$ the equivalence between restricted weak-type and (1) breaks down, however there is an acceptable substitute.

Lemma 3.1. *Suppose* $1 \leq p \leq \infty$ *and* $0 < q \leq \infty$*, and* T *is of restricted weak-type* (p, q)*. Then for every pair of integers* k, l *and all* $f_k \in F_k(X)$*,* $g_l \in F_l(Y)$*, we can find* $g_l' \in F_{l-1}(Y)$ *such that*

$$|\Lambda(f_k, g_l - g_l')| \lesssim 2^{k/p} 2^{l/q'}. \tag{3}$$

If $q > 1$ then one could simply telescope this to obtain (1), but for $q \leq 1$ this is not possible as q' is non-positive. The implication is not quite reversible for $q < 1$, but it is very close. (In practice, the definition (3) seems to be a better notion to use than restricted weak-type in any event).

Proof Let f_k, g_l be as above. Since $\|Tf_k\|_{q,\infty} \lesssim \|f_k\|_p \lesssim 2^{k/p}$ by hypothesis, we have

$$|\{|Tf_k| > C2^{-l/q}2^{k/p}\}| \leq \frac{1}{2}2^l$$

for sufficiently large C. The claim then follows by letting g_l' be the restriction of g_l to the above set. ■

Using this, we can now interpolate below L^1. A representative result is as follows.

Proposition 3.2. *Let T be an operator such that both T and T^* is of weak-type (1,1). Then T is bounded on L^p for every $1 < p < \infty$.*

Proof It suffices to show that T is of restricted weak-type (p,p) for all such p, i.e. that

$$|\Lambda(f_k, g_l)| \lesssim 2^{k/p}2^{l/p'}$$

for all $f_k \in F_k(X)$, $g_l \in F_l(Y)$.

Fix p, and let A be the best constant such that

$$|\Lambda(f_k, g_l)| \leq A2^{k/p}2^{l/p'} \tag{4}$$

for all k, l, f_k, g_l. By our a priori assumption A is finite.

Now suppose $k \leq l$, so $2^k \leq 2^{k/p}2^{l/p'}$. Since T is of restricted weak-type (1,1), we can find g_{l-1} such that

$$|\Lambda(f_k, g_l - g_{l-1})| \lesssim 2^k \leq C2^{k/p}2^{l/p'}$$

Combining this with (4) with g_l replaced by g_{l-1}, we obtain

$$|\Lambda(f_k, g_l)| \leq (A2^{-1/p'} + C)2^{k/p}2^{l/p'}.$$

Applying the same type of reasoning when $k > l$, but using T^* instead of T, we see that the above estimate holds for all k, l. Taking suprema we obtain

$$A \leq A2^{-1/p'} + C.$$

Thus A is finite, and the claim follows. ■

Thanks to Loukas Grafakos for helpful comments.

References

[1] M. Keel, T. Tao, *Endpoint Strichartz Estimates*, Amer. Math. J. 120 (1998), 955–980.

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