THE BANACH-TARSKI PARADOX

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ABSTRACT. An exposition of the Banach-Tarski paradox.

1. INTRODUCTION

Let $SO(3)$ denote the group of rotation operators on $\mathbf{R}^3$. The first version of the Banach-Tarski paradox, which we give below, shows that we can find a group of rotations with the curious property that G can be disassembled into four pieces, which after rotation can be reassembled to form two complete copies of G.

Theorem 1.1 (Banach-Tarski paradox, first version)**.** *There exists a countable subgroup G of $SO(3)$, and a partition*

$$G = G_1 \uplus G_2 \uplus G_3 \uplus G_4$$

into disjoint sets G_1, G_2, G_3, G_4, such that one can write

$$G = G_1 \uplus AG_2 = G_3 \uplus BG_4$$

for some rotations $A, B \in SO(3)$.

Proof We first need to find two rotation operators A, B such that no non-trivial word from the alphabet A, B, A^{-1}, B^{-1} gives the identity, where "nontrivial" means that the word is non-empty, that A and A^{-1} are never adjacent, and B and B^{-1} are never adjacent. This is easily done but requires some algebra and we defer it to the Appendix. Assuming we have these operators, we let G be the subgroup of $SO(3)$ generated by A, B, A^{-1}, B^{-1}; this is thus the group of all non-trivial words using the alphabet A, B, A^{-1}, B^{-1}, together with the empty word I, which is the identity. In particular we have the partition

$$G = \{I\} \cup G(A) \cup G(A^{-1}) \cup G(B) \cup G(B^{-1})$$

where $G(A)$ is the set of all non-trivial words in G which start with A, etc. Observe that

$$G = G(A) \uplus AG(A^{-1}),$$

because every word in G either starts with A, or is equal to A times a non-trivial word beginning with A^{-1}, but never both. Similarly we have

$$G = G(B) \uplus BG(B^{-1}).$$

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We are almost done, except that we have to deal with the empty word I. But this is easily accomplished by defining

$$G_1 := G(A) \cup \{I, A^{-1}, A^{-2}, A^{-3}, \dots\}$$
$$G_2 := G(A^{-1}) \backslash \{A^{-1}, A^{-2}, A^{-3}, \dots\}$$
$$G_3 := G(B)$$
$$G_4 := G(B^{-1})$$

and one easily verifies that all the claims of the theorem hold. ∎

Note that the above theorem did not require the axiom of choice. However, the following corollary, which passes from the countable rotation group G to most of the sphere S^2, does rely very much on the axiom of choice.

Corollary 1.2 (Hausdorff paradox, first version)**.** *There exists a countable subset C of the sphere S^2, and a decomposition*

$$(S^2 \backslash C) = \Omega_1 \uplus \Omega_2 \uplus \Omega_3 \uplus \Omega_4$$

such that

$$(S^2 \backslash C) = \Omega_1 \uplus A\Omega_2 = \Omega_3 \uplus B\Omega_4$$

for some rotation matrices $A, B \in SO(3)$.

Proof Let $A, B, G, G_1, G_2, G_3, G_4$ be as in Theorem 1.1. Each rotation in G fixes two points on the sphere S^2 (the intersection of S^2 with the axis of the rotation); let C be the union of all those fixed points. Then the rotation group G acts freely on the complement $S^2 \backslash C$. Thus, using the axiom of choice, one can foliate $S^2 \backslash C = \biguplus_{x \in X} Gx$ for some set X (picking one representative from each G-orbit in $S^2 \backslash C$). If one then sets $\Omega_i := \biguplus_{x \in X} G_i x$ then the claim now follows from Theorem 1.1. ∎

Next, we eliminate this countable set C, using the following simple lemma.

Lemma 1.3. *Let C be a countable subset of the sphere S^2. Then there exists a decomposition*

$$S^2 = \Sigma_1 \uplus \Sigma_2$$

such that

$$S^2 \backslash C = \Sigma_1 \uplus R\Sigma_2$$

for some rotation matrix $R \in SO(3)$.

Proof Pick R at random. Since C is countable, then with probability 1 we can ensure that any two elements of C lie in distinct R-orbits, i.e. $R^i C \cap R^j C = \emptyset$ whenever $i \neq j$. We then set

$$\Sigma_2 := C \cup RC \cup R^2 C \cup \dots; \quad \Sigma_1 := S^2 \backslash \Sigma_2$$

and the claim follows. ∎

Combining this Lemma with the preceding corollary, we obtain

Corollary 1.4. *There exists a partition*

$$S^2 = \Gamma_1 \uplus \dots \uplus \Gamma_8$$

and rotation matrices $R_1, \ldots, R_8 \in SO(3)$ *such that*

$$S^2 = \biguplus_{i=1}^{4} R_i \Gamma_i = \biguplus_{i=5}^{8} R_i \Gamma_i.$$

Since the punctured ball $B^3 \backslash 0$ can be viewed in polar co-ordinates as the product of the sphere S^2 and the interval $(0, 1]$, we conclude

Corollary 1.5 (Banach-Tarski paradox). *There exists a partition*

$$B^3 \backslash \{0\} = E_1 \uplus \ldots \uplus E_8$$

and rotation matrices $R_1, \ldots, R_8 \in SO(3)$ *such that*

$$B^3 \backslash \{0\} = \biguplus_{i=1}^{4} R_i E_i = \biguplus_{i=5}^{8} R_i E_i.$$

Of course, at least one of the E_i has to be non-Lebesgue measurable. One can eliminate the puncture at the origin if one allows translations as well as rotations, by using a trick similar to that used to prove Lemma 1.3; we leave this as an exercise to the reader.

2. Appendix: the algebraic bit

We now give the rotation operators $A, B \in SO(3)$ needed to prove Theorem 1.1. It turns out that we can give very explicit matrices, namely

$$A(x, y, z) := (\frac{3}{5}x + \frac{4}{5}y, -\frac{4}{5}x + \frac{3}{5}y, z); \quad B(x, y, z) := (x, \frac{3}{5}y + \frac{4}{5}z, -\frac{4}{5}y + \frac{3}{5}z).$$

These are easily seen to be rotation matrices with inverses

$$A^{-1}(x, y, z) := (\frac{3}{5}x - \frac{4}{5}y, \frac{4}{5}x + \frac{3}{5}y, z); \quad B^{-1}(x, y, z) := (x, \frac{3}{5}y - \frac{4}{5}z, \frac{4}{5}y + \frac{3}{5}z).$$

Now we claim that no non-trivial composition of A, B, A^{-1}, B^{-1} gives the identity. It suffices to show that no non-trivial composition of the operators $5A$, $5B$, $5A^{-1}$, $5B^{-1}$ gives a linear operator whose coefficients are all divisible by 5. We now work in the finite field geometry F_5^3, where $F_5 = \mathbf{Z}/5\mathbf{Z}$ is the field of order 5. Then we have

$$5A(x, y, z) := (3x + 4y, -4x + 3y, 0); \quad 5B(x, y, z) := (0, 3y + 4z, -4y + 3z)$$

and

$$5A^{-1}(x, y, z) := (3x - 4y, 4x + 3y, 0); \quad 5B^{-1}(x, y, z) := (0, 3y - 4z, 4y + 3z).$$

Each of these operators are rank one operators in F_5^3:

$$\begin{aligned} \text{range}(5A) &= \text{span}((3, -4, 0)) = \ker(5A^{-1})^{\perp} \\ \text{range}(5A^{-1}) &= \text{span}((3, 4, 0)) = \ker(5A)^{\perp} \\ \text{range}(5B) &= \text{span}((0, 3, -4)) = \ker(5B^{-1})^{\perp} \\ \text{range}(5B^{-1}) &= \text{span}((0, 3, 4)) = \ker(5B)^{\perp}. \end{aligned}$$

From this we see that any non-trivial combination of $5A$, $5A^{-1}$, $5B$, $5B^{-1}$ (in which $5A$ and $5A^{-1}$ are never adjacent, and $5B$ and $5B^{-1}$ are never adjacent) will always be a non-zero operator, as desired, because the ranges and kernels are skew.

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