A HASTY INTERPOLATION

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Abstract. A cautionary note about using interpolation incorrectly.

1. Introduction

Let $f_0, \dots, f_{n-1}$ be functions on a measure space X which are orthogonal. From Pythagoras' theorem we have

$$\|f_0 + \dots + f_{n-1}\|_2 = (\sum_{k=0}^{n-1} \|f_k\|_2^2)^{1/2}$$

whereas from the triangle inequality we have

$$\|f_0 + \dots + f_{n-1}\|_1 \leq \sum_{k=0}^{n-1} \|f_k\|_1$$

One may thus be led to conjecture the interpolated inequality

$$\|f_0 + \dots + f_{n-1}\|_p \leq (\sum_{k=0}^{n-1} \|f_k\|_p^p)^{1/p} \tag{1}$$

for all $1 < p < 2$. However, such estimates are false, even when $n = 2$. An example is

$$f_0 = \chi_{[-N^2,0]} + N\chi_{[0,1]}, \quad f_1 = \chi_{[-N^2,0]} - N\chi_{[0,1]};$$

we have

$$\|f_0 + f_1\|_p = 2N^{2/p}, \quad (\|f_0\|_p^p + \|f_1\|_p^p)^{1/p} = 2^{1/p}(N^2 + N^p)^{1/p},$$

and so the right-hand side of (1) is larger than the left for sufficiently large N. By iterating this example, e.g. using the four functions $f_i \otimes f_j$ for $i, j = 0, 1$, we can make the ratio between the two sides of (1) arbitrarily large.

A similar naive interpolation would yield

$$\|f_0 + \dots + f_{n-1}\|_p \leq (\sum_{k=0}^{n-1} \|f_k\|_p^{p'})^{1/p'} \tag{2}$$

for $2 < p < \infty$, but this is also false using the same example:

$$\|f_0 + f_1\|_p = 2N^{2/p}, \quad (\|f_0\|_p^p + \|f_1\|_p^{p'})^{1/p'} = 2^{1/p'}(N^2 + N^p)^{1/p},$$

for sufficiently *small* N.

The problem with both of these "interpolations" is that the inequality

$$\|f_0 + \dots + f_{n-1}\|_2 \leq (\sum_{k=0}^{n-1} \|f_k\|_2^2)^{1/2}$$

does not hold for all f_j ranging over Banach spaces, but is subject to a constraint which is not amenable to interpolation.

However, substitutes exist when the orthogonality is made more robust. For instance, both (1) and (2) hold under the additional assumption that the $|f_j|$ are constant multiples of characteristic functions (why?). Also, we have

Lemma 1.1. [1] *Let* R_k *be a collection of rectangles in frequency space such that the dilates* $2R_k$ *are almost disjoint, and suppose that* f_k *are a collection of functions whose Fourier transforms are supported on* R_k*. Then for all* $1 \leq p \leq \infty$ *we have*

$$(\sum_k \|f_k\|_p^{p^*})^{1/p^*} \lesssim \|\sum_k f_k\|_p \lesssim (\sum_k \|f_k\|_p^{p_*})^{1/p_*},$$

where $p_* = \min(p, p')$*,* $p^* = \max(p, p')$*.*

References

[1] T. Tao, A. Vargas, *A bilinear approach to cone multipliers II. Applications*, to appear, GAFA.

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