Mathematics 245B
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Instructions: Try to do all three problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ______________________

Nickname: ______________________

Student ID: ______________________

Signature: ______________________

Problem 1. ____________

Problem 2. ____________

Problem 3. ____________

Total: ______________________

Problem 1. Let $f : \mathbf{R} \to \mathbf{R}$ be an absolutely integrable function. For any $t > 0$, let $P_t f(x)$ denote the function

$$P_t f(x) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{(y-x)^2 + t^2} f(y)\ dy;$$

this is known as the *harmonic extension* of f and is of importance in the theory of harmonic functions. Establish the inequality $\sup_{t>0} |P_t f(x)| \leq CMf(x)$ for all $x \in \mathbf{R}$, where $Mf(x)$ is the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\ dy$$

and $0 < C < \infty$ is an absolute constant (not depending on f or x).

(Hint: split the region of integration for $P_t f(x)$ into the interval $\{y : |y - x| \leq t\}$ and to the sets $\{y : 2^j t < |y - x| \leq 2^{j+1} t\}$ for $j = 0, 1, 2, \ldots$. Estimate the integral on each set by some multiple of $Mf(x)$, and then sum in j. This argument should give a constant C which looks something like

$$C := \frac{2}{\pi} + \sum_{j=0}^{\infty} \frac{1}{\pi} \frac{2^{j+2}}{1 + 2^{2j}} \approx 2.5.$$

The best value of C that one can get is in fact $C = 1$, but you are not required to obtain this sharp result.)

Fix $x \in \mathbf{R}$ and $t > 0$. By the triangle inequality we have

$$\pi |P_t f(x)| \leq \int_{-\infty}^{\infty} \frac{t}{(y-x)^2 + t^2} |f(y)|\ dy$$

(noting that $\frac{t}{(y-x)^2+t^2}$ is positive). Let $E_0 := \{y : |y - x| \leq t\}$ and $F_j := \{\{y : 2^j t < |y - x| \leq 2^{j+1} t\}$ for $j = 0, 1, 2, \ldots$. It is clear that E_0 and $F_0, F_1, \ldots$ are disjoint, and their union is equal to $\mathbf{R}$. Thus (using dominated convergence or monotone convergence)

$$\pi |P_t f(x)| \leq \frac{1}{\pi} \int_{E_0} \frac{t}{(y-x)^2 + t^2} |f(y)|\ dy + \sum_{j=0}^{\infty} \frac{1}{\pi} \int_{F_j} \frac{t}{(y-x)^2 + t^2} |f(y)|\ dy.$$

On the set E_0, $\frac{t}{(y-x)^2+t^2}$ is at most $\frac{t}{t^2} = \frac{1}{t}$, and hence

$$\int_{E_0} \frac{t}{(y-x)^2 + t^2} |f(y)|\ dy \leq \int_{x-t}^{x+t} \frac{1}{t} |f(y)|\ dy = 2 \frac{1}{2t} \int_{x-t}^{x+t} |f(y)|\ dy \leq 2Mf(x)$$

by definition of $Mf(x)$. Similarly, on the set F_j, $\frac{t}{(y-x)^2+t^2}$ is at most $\frac{t}{(2^j t)^2 + t^2} = \frac{1}{(2^{2j}+1)t}$, and F_j is contained inside the interval $[x - 2^{j+1} t, 2^{j+1} t]$, thus

$$\int_{F_j} \frac{t}{(y-x)^2 + t^2} |f(y)|\, dy \leq \int_{x-2^{j+1}t}^{x+2^{j+1}t} \frac{1}{t(2^{2j}+1)} |f(y)|\, dy = \frac{2^{j+2}}{(2^{2j}+1)} \int_{2 \times 2^{j+1} t} \int_{x-2^{j+1}t}^{x+2^{j+1}t} |f(y)|\, dy \leq \frac{2^{j+2}}{(2^{2j}+1)} Mf(x).$$

Summing this we obtain

$$\pi|P_t f(x)| \leq 2Mf(x) + \sum_{j=0}^{\infty} \frac{2^{j+2}}{(2^{2j}+1)} Mf(x)$$

and thus $|P_t f(x)| \leq CMf(x)$, where C is the constant described above. Taking suprema over all t one obtains the result.

To get the sharp constant $C = 1$, one argues as follows. For $y \in \mathbf{R}$, let $F(y) := \int_{-\infty}^{y} |f(z)|\, dz$. Then by the Fundamental theorem of calculus, F is absolutely continuous and bounded and $|f(y)| = F'(y)$ almost everywhere. Thus

$$\pi|P_t f(x)| \leq \int_{-\infty}^{\infty} \frac{t}{(y-x)^2+t^2} |f(y)|\, dy = \int_{x}^{\infty} \frac{t}{(y-x)^2+t^2} F'(y)\, dy.$$

We may integrate by parts to write this as

$$\int_{-\infty}^{\infty} \frac{2(y-x)t}{((y-x)^2+t^2)^2} F(y)\, dy.$$

(I'll leave it to you to justify why this integration by parts is rigorous despite the interval of integration being infinite. Hint: restrict to an interval $[-N, N]$ for some large N, and take limits as $N \to \infty$ exploiting the fact that F is bounded). We split into $y \geq x$ and $y \leq x$, and change variables to $y = x + r$ or $y = x - r$ to obtain

$$\int_{0}^{\infty} \frac{2rt}{(r^2+t^2)^2} F(x+r)\, dr - \int_{0}^{\infty} \frac{2rt}{(r^2+t^2)^2} F(x-r)\, dr.$$

On the other hand, we have

$$F(x+r) - F(x-r) = \int_{x-r}^{x+r} |f(z)|\, dz \leq 2rMf(x)$$

by definition of $Mf(x)$. Thus we have

$$\pi|P_t f(x)| \leq \int_{0}^{\infty} \frac{2rt}{(r^2+t^2)^2} 2rMf(x)\, dr.$$

An elementary exercise establishes that

$$\int_{0}^{\infty} \frac{2rt}{(r^2+t^2)^2} 2r = \pi$$

and so we have $|P_t f(x)| \leq Mf(x)$ for all $t > 0$. Taking suprema over all t we obtain the claim.

As a side remark - by repeating the proof of the Lebesgue differentiation theorem one can also prove that $\lim_{t \to 0} P_t f(x) = f(x)$ for almost every x. This is known as Fatou's theorem.

Problem 2. (a) Let X be a locally compact Hausdorff space, and let Y be a closed subset of X. Show that Y is also locally compact Hausdorff.

First observe that any subspace of a Hausdorff space is still Hausdorff (if two open sets in X separate two points in Y, then their restriction to Y will be open in Y, and continue to separate those two points). So it suffices to verify local compactness. Let $y \in Y$, then by hypothesis there is a compact (hence closed) neighbourhood K of y in X, which contains an open neighbourhood V of y. The set $K \cap Y$ is a closed subset of the compact set K, and is hence compact, hence is compact in Y (compactness is intrinsic). It contains $V \cap Y$, which is an open neighbourhood of y in Y. Thus $K \cap Y$ is a compact neighbourhood of y in Y, and the claim follows.

(b) Let X be a locally compact Hausdorff space, and let Y be an open subset of X. Show that Y is also locally compact Hausdorff.

As before it suffices to verify the local compactness property. Let $y \in Y$, then by Proposition 4.30 there exists a compact neighbourhood N of y in X which is contained in Y. Since compactness is intrinsic, N is also a neighbourhood of y in Y, and the claim follows.

Problem 3. Let X be a non-empty topological space, and let $C(X)$ be the space of real-valued continuous functions on X. Show that the set

$$\{E \subseteq X : 1_E \in C(X)\}$$

has cardinality two if and only if X is connected. (Recall that $1_E : X \to \mathbf{R}$ is the indicator function of E, with $1_E(x) = 1$ when $x \in E$ and $1_E(x) = 0$ when $x \notin E$).

Note that if E is empty or equal to X, then 1_E is constant and hence clearly continuous. Thus the set $\{E \subseteq X : 1_E \in C(X)\}$ always has at least two elements (recall that X is assumed to be non-empty).

Now suppose that X is connected. Suppose for contradiction that there existed an E not equal to $\emptyset$ or X such that $1_E \in C(X)$. Then $1_E^{-1}(\{0\}) = X \backslash E$ and $1_E^{-1}(\{1\}) = E$ are inverse images of closed sets via a continuous function and are hence closed. But then X is partitioned into two disjoint non-empty closed (hence open) sets, and is therefore disconnected, a contradiction.

Conversely, suppose that $\{E \subseteq X : 1_E \in C(X)\}$ contained no sets other than $\emptyset$ and X. Suppose for contradiction that X is disconnected, thus $X = E \cap F$ for some disjoint open non-empty sets E, F. Consider the function 1_E. For any set V in $\mathbf{R}$ (open or otherwise), the set $1_E^{-1}(V)$ is equal to $\emptyset$, E, F, or X - all of which are open. Thus 1_E is continuous, and so $\{E \subseteq X : 1_E \in C(X)\}$ has cardinality at least three, a contradiction.

Remark: Another amusing way to phrase this problem is that the number of idempotents in the algebra $C(X)$ (i.e. elements $f \in C(X)$ such that $f^2 = f$) is always at least two, and is equal to two precisely when X is connected.

Extra challenge: try showing that if the set $\{E \subseteq X : 1_E \in C(X)\}$ is finite, then the cardinality is always a power of two.
