Mathematics 245A
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Midterm, Nov 3, 2004

Instructions: Try to do all three problems; they are all of equal value. There is plenty of working space, and a blank page at the end. Throughout this midterm, "measure" refers to a non-negative, countably additive measure. You may use the axiom of choice freely.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ____________________

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Student ID: ____________________

Signature: ____________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Total: ____________________

Problem 1. Let X be a set, let $\mathcal{E}$ be a collection of subsets of X, and let $\mathcal{M}(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E}$. Let μ, ν be two measures on $\mathcal{M}(\mathcal{E})$ such that $\mu(X) = \nu(X) < \infty$, and are such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{E}$. Prove that in fact we have $\mu(A) = \nu(A)$ for all $A \in \mathcal{M}(\mathcal{E})$. (Hint: what can you say about the set $\mathcal{C} := \{A \in \mathcal{M}(\mathcal{E}) : \mu(A) = \nu(A)\}$?)

Unfortunately, this statement is false. Here is a counterexample: Let $X = \{a, b, c, d\}$ be a four-element set, let $\mathcal{E} = \{\{a, b\}, \{b, c\}\}$, then it is easy to see that $\mathcal{M}(\mathcal{E})$ is the power set of X. Let μ be the measure such that $\mu(\{a\}) = \mu(\{c\}) = 1$ and $\mu(\{b\}) = \mu(\{d\}) = 0$, and let ν be the measure such that $\nu(\{a\}) = \nu(\{c\}) = 0$ and $\nu(\{b\}) = \nu(\{d\}) = 1$. Then μ and ν agree on $\mathcal{E}$ but do not agree on $\mathcal{M}(\mathcal{E})$.

(My intention with this question was for you to show that $\mathcal{C}$ was a σ-algebra that contains $\mathcal{E}$, and hence must necessarily contain $\mathcal{M}(\mathcal{E})$. Unfortunately, while $\mathcal{C}$ does indeed contain $\emptyset$ and X, and is closed under complementations, disjoint unions, and closed under countable increasing unions and countable decreasing intersections), it is not closed under finite intersections or finite unions. For instance, in the above example $\mathcal{C} = \{\emptyset, \{a, b\}, \{b, c\}, \{a, d\}, \{c, d\}, X\}$.)

The statement is true however if $\mathcal{E}$ is an algebra, basically because of Theorem 1.14. If one lets μ_0 be the restriction of μ (or ν) to $\mathcal{E}$, then μ_0 is a premeasure (why?), and hence by Theorem 1.14 there is a unique extension of μ_0 to $\mathcal{M}(\mathcal{E})$ which is a measure, and thus μ and ν must both equal this extension, and are thus equal to each other.

The statement is also true under the weaker assumption that $\mathcal{E}$ is an elementary family, because μ and ν will then also agree on the space of finite disjoint unions of sets in $\mathcal{E}$, which is an algebra.

Problem 2. (a) Let $A \subseteq \mathbf{R}$ be an arbitrary subset of the real line (not necessarily Lebesgue measurable), and let $\mu^*(A)$ denote the Lebesgue outer measure of A. Show that there exists a Lebesgue measurable set $B \subseteq \mathbf{R}$ such that $A \subseteq B$ and $\mu(B) = \mu^*(A)$.

Let $n \geq 1$, then by definition of outer measure, there exists a countable family of intervals $(I_j^n)_{j=1}^\infty$ which covers A and is such that

$$\sum_{j=1}^{\infty} \mu(I_j^n) \leq \mu^*(A) + \frac{1}{n}.$$

By countable sub-additivity, this means that

$$\mu(\bigcup_{j=1}^{\infty} I_j^n) \leq \mu^*(A) + \frac{1}{n}.$$

Thus if we let $B := \bigcap_{n=1}^\infty (\bigcup_{j=1}^\infty I_j^n)$, then B is measurable (countable intersection of countable union of intervals) and covers A, and $\mu(B) \leq \mu^*(A) + \frac{1}{n}$ for all $n \geq 1$, hence $\mu(B) \leq \mu^*(A)$. On the other hand, by monotonicity of outer measure $\mu(B) = \mu^*(B) \geq \mu^*(A)$. Thus $\mu(B) = \mu^*(A)$ as claimed.

(b) Prove that Lebesgue outer measure is continuous from below. In other words, if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \subset \mathbf{R}$, prove that $\mu^*(\bigcup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu^*(A_n)$. (Note: the argument I gave in class was incorrect. However, you may proceed by using (a)).

For each A_n, let us choose a measurable B_n such that $\mu(B_n) = \mu^*(A_n)$ and B_n contains A_n. Now let $C_n := \bigcap_{m=n}^\infty B_m$, then C_n is still measurable and still contains A_n (because each B_m contains A_m and hence contains A_n when $m \geq n$). Also

$$\mu^*(A_n) \leq \mu^*(C_n) = \mu(C_n) \leq \mu(B_n) = \mu^*(A_n)$$

and hence $\mu(C_n) = \mu^*(A_n)$ for all n. Since the C_n are increasing, we see using continuity from below that

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \mu^*(\bigcup_{n=1}^{\infty} C_n) = \mu(\bigcup_{n=1}^{\infty} C_n) = \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \mu^*(A_n),$$

so

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \lim_{n\to\infty} \mu^*(A_n).$$

On the other hand, by monotonicity we have

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \geq \mu^*(A_n)$$

for all n, hence

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \lim_{n\to\infty} \mu^*(A_n).$$

On the other hand, since $\bigcup_{n=1}^{\infty} A_n$ containws each n, we have

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \geq \mu^*(A_n)$$

for each n, and hence

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \geq \lim_{n\to\infty} \mu^*(A_n).$$

Combining these two, we obtain

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu^*(A_n)$$

as desired.

Note that this argument shows in fact that any outer measure arising from a premeasure via Caratheodory's construction will necessarily be continuous from below. Note also that the trick of passing from the B_n (which are not nested) to the C_n (which are) is quite crucial.

Problem 3. Let $f : \mathbf{R} \to \mathbf{R}$ be a Lebesgue measurable function (i.e. it is $(\mathcal{L}, \mathcal{B}_R)$-measurable). Let E denote the set of all points x in $\mathbf{R}$ at which f is differentiable, i.e.

$$E := \{x \in \mathbf{R} : \lim_{h \to 0; h \neq 0} \frac{f(x+h) - f(x)}{h} \text{ exists}\}.$$

Prove that E is Lebesgue measurable.

The intended solution to this problem was to use the fact that the functions $\frac{f(x+h)-f(x)}{h}$ were measurable for each h. Unfortunately the fact that there are uncountably many h involved causes significant problems with this approach. The following alternative approach works, in fact it works for all functions f, not just the Lebesgue measurable ones, and it makes the slightly stronger assertion that E is a Borel set, not just a Lebesgue measurable set.

Let us first define the sets $E_{a,b,\varepsilon}$ for $a < b$ and $\varepsilon > 0$ by

$$E_{a,b,\varepsilon} := \{x \in \mathbf{R} : a \leq \frac{f(x+h) - f(x)}{h} \leq b \text{ for all } 0 < |h| < \varepsilon\}.$$

We first clam that $E_{a,b,\varepsilon}$ is always closed, thus that $E_{a,b,\varepsilon}$ contains all its right limit points and left limit points. To see this, suppose that there is a sequence $(x_n)_{n=1}^\infty$ in $E_{a,b,\varepsilon}$ which converges to some real number x from either the right or the left; we need to show that x also lies in $E_{a,b,\varepsilon}$. Without loss of generality we assume that x_n converges to x from the left, thus $x_n < x$ for all n. For n large enough, we have $0 < x - x_n < \varepsilon$; since $x_n \in E_{a,b,\varepsilon}$, this implies

$$a \leq \frac{f(x) - f(x_n)}{x - x_n} \leq b \text{ for all sufficiently large } n.$$

Multiplying this by $x - x_n$ and then adding $f(x_n)$, we obtain

$$f(x_n) + a(x - x_n) \leq f(x) \leq f(x_n) + b(x - x_n) \text{ for all sufficiently large } n.$$

Applying the squeeze test, we see that $f(x_n)$ converges to $f(x)$ as $n \to \infty$.

Now let $0 < |h| < \varepsilon$. We write

$$\frac{f(x+h) - f(x)}{h} = \lim_{n \to \infty} \frac{f(x_n + (h + x - x_n)) - f(x_n)}{h} + \frac{f(x_n) - f(x)}{h}.$$

The second sum goes to zero as $n \to \infty$ (recall that h is fixed). Also, for n sufficiently large, we have $0 < |h + x - x_n| < \varepsilon$, and hence (since $x_n \in E_{a,b,\varepsilon}$)

$$a \leq \frac{f(x_n + (h + x - x_n)) - f(x_n)}{h} \leq b.$$

As a consequence we have

$$a \leq \frac{f(x+h) - f(x)}{h} \leq b.$$

Since $0 < |h| < \varepsilon$ was arbitrary, we see that $x \in E_{a,b,\varepsilon}$ as claimed.

Let

$$\tilde{E}_{a,b,\varepsilon} := \{x \in \mathbf{R} : a < \frac{f(x+h)-f(x)}{h} < b \text{ for all } 0 < |h| < \varepsilon\}.$$

It is easy to see that each $\tilde{E}_{a,b,\varepsilon}$ is a countable union of sets of the form $E_{a+1/n,b-1/n,\varepsilon}$ and is thus Borel (in fact it is F_σ). Now let

$$\tilde{E}_{a,b} := \{x \in \mathbf{R} : a < \liminf_{h \to 0; h \neq 0} \frac{f(x+h)-f(x)}{h} \leq \limsup_{h \to 0; h \neq 0} \frac{f(x+h)-f(x)}{h} < b\}.$$

It is easy to see that this set is a countable union of sets of the form $\tilde{E}_{a,b,1/n}$ and is thus Borel (in fact it is F_σ). Now for each $r > 0$, let

$$\tilde{E}_r := \{x \in \mathbf{R} : q-r < \liminf_{h \to 0; h \neq 0} \frac{f(x+h)-f(x)}{h} \leq \limsup_{h \to 0; h \neq 0} \frac{f(x+h)-f(x)}{h} < q \text{ for some rational } q\}.$$

It is easy to see that each E_r is a countable union of sets of the form $\tilde{E}_{q-r,q}$ and is thus Borel (in fact it is F_σ). Now we observe that E is the countable intersection of sets of the form $\tilde{E}_{1/n}$ and is thus Borel (in fact it is $F_{\sigma\delta}$).