Embedding a snowflake metric space into Euclidean space

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joint work with Jim Skon and Preston Pennington

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The project

General question: How can we best represent a metric space with Euclidean coordinates?

There are metric spaces that do not embed bi-Lipschitzly in any Euclidean space. However, if the metric space is doubling, then Assouad’s theorem guarantees that every snowflake of the space does embed bi-Lipschitzly in some $\mathbb{R}^n$. 
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A classical example of doubling metric space with no bi-Lip embedding into $\mathbb{R}^n$ constructed by Laakso.

We are interested in embedding the snowflaked Laakso space into $\mathbb{R}^n$. This is joint work with Jim Skon (Kenyon CS) and Preston Pennington (Kenyon ’20).
Graph Metric Spaces

Recall that a *metric space* is a set $X$ with a distance function $d : X \times X \to \mathbb{R}^+$ that satisfies

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

for all points $x, y, z$ in $X$.

**Graph metric spaces:** Given graph $(V, E)$, define distance on $V$ so that $d(x, y)$ is the length ($\#$ edges) of the shortest path between vertices $x$ and $y$.

Can also assign positive weights to the edges for non-integer distances.
Doubling Metric Spaces

A metric space \((X, d)\) is *doubling* if there exists a constant \(C \geq 1\) so that every ball of radius \(r\) can be “covered by” at most \(C\) balls of radius at most \(r/2\).
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Examples

- \(\mathbb{R}^n\) and subsets of \(\mathbb{R}^n\), for all \(n\): Doubling
- The following infinite graph with the path metric: Not Doubling
The Laakso Space (as simplified by Lang and Plaut)

Construct as the limit of a sequence of graphs

length 1
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Construct as the limit of a sequence of graphs
Metric Space Embeddings

A map $f : X \to Y$ is an embedding if it is a homeomorphism onto its image.

**Competing goals:**

Find an embedding into the simplest (lowest dimensional) space possible!

Also look for an embedding that doesn’t distort the metric too much!

- **Isometry:** distances preserved exactly

  $d(x, y) = d(f(x), f(y))$

- **Bi-Lipschitz map:** distances distorted by a bounded amount

  $\frac{1}{L} \cdot d(x, y) \leq d(f(x), f(y)) \leq L \cdot d(x, y)$
Metric Space Embeddings $f : X \rightarrow \mathbb{R}^n$

Isometric embeddings into $\mathbb{R}^n$: too much to hope for.
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**Isometric embeddings into** $\mathbb{R}^n$: too much to hope for.

**bi-Lipschitz embeddings into** $\mathbb{R}^n$:

Doubling is necessary (doubling property is bi-Lip invariant).

Doubling is not sufficient: Shown by Semmes, 1996; simpler example by Laakso, 2002.
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Snowflaking a Metric Space

Given a metric space \((X, d)\) and \(\alpha \in (0, 1]\), set

\[d^\alpha(x, y) := (d(x, y))^\alpha.\]

\((X, d^\alpha)\) is a metric space, called the \(\alpha\)-snowflake of \((X, d)\).
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Why do we call it snowflaking?

\[
[0, 1]^\alpha \xrightarrow{bi-Lip} \mathbb{R}^2, \quad \alpha = \log 3 / \log 4
\]
Assouad’s Theorem

Theorem (Assouad, 1983)

Each snowflaked version of a doubling metric space admits a bi-Lipschitz embedding in some Euclidean space. In particular, the distortion $L$ of the embedding and dimension $N$ of the target space each depend on both the snowflaking constant and on the doubling constant.
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Theorem (Naor-Neiman, 2012)

For snowflaking constants $\alpha \in (1/2, 1)$, the dimension $N$ can be chosen independent of the snowflaking constant!
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Theorem (Naor-Neiman, 2012)

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Key ingredients in the proof: random embeddings at different scales and a version of the “Lovász local lemma.”
An improvement to Assouad’s Theorem

Non-Probabilistic Proof (David-Snipes, 2013).
Big picture idea of the construction:

- Choose a sequence of scales $r_k$ (powers of a small parameter $\tau$).
- For each scale choose a maximal $r_k$-separated set of “grid points” in the metric space.
- Color the grid points at every level.
- Define the embedding based on the colorings of all the grid points.
- Scales $\leftrightarrow$ digits, and colors $\leftrightarrow$ coordinate directions (coordinate subspaces) of Euclidean space.
Embedding the Snowflaked Laakso Space

Assign an address (signature) to each point:
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![Diagram showing embedding of points with coordinates]

Coordinates shown:
- \( <0, 0, 2> \)
- \( <0, 0, 3> \)
- \( <0, 0, 1> \)
- \( <0, 0, -2> \)
- \( <0, 1, 0> \)
- \( <0, -2, 0> \)
- \( <1, 0, 0> \)
- \( <0, 2, 0> \)
- \( <0, 3, 0> \)
Embedding the Snowflaked Laakso Space

Assign an address (signature) to each point:
Embedding the Snowflaked Laakso Space

- Choose constants
  - Snowflaking constant \( \alpha > 2/3 \): Set \( \alpha = \log 3 / \log 4 \).
  - Small parameter \( \tau < 1 - \alpha \) that gives a sequence of scales: Set \( \tau = 1/64 \); then scales are \( r_k = \tau^{2^k} \).

- For each scale \( r_k \), choose a maximal \( r_k \)-separated set of “grid points” in the metric space.
  - Since \( \tau = 1/4^3 \), the \( k \)th set of grid points is just the \( 6k \)th stage in the construction of the space.

- Color the grid points at every level. No two points within \( 10r_k \) of each other can share the same color.
Coloring the $r_k$-separated sets

Greedy algorithm:
- Enumerate the set of colors
- Enumerate the set of grid points
- Each grid point gets smallest possible color

A priori, number of colors needed is large: $C^5 = 6^5 = 7776$. 
Coloring the \( r_k \)-separated sets

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We implemented this coloring algorithm and found that the maximum number of colors needed is just 31.

Problem: Greedy is expensive!
Coloring the $r_k$-separated sets

A smarter algorithm (Pennington, 2018):
Based on stage 3

Total of 36 colors.

The two colorings can be appended to themselves or each other without violating proximity rules.
Coloring the $r_k$-separated sets

A smarter algorithm (Pennington, 2018): Combine 2 colorings for Stage 3 to color any higher level.

Level 4:
Coloring the $r_k$-separated sets

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Level 5:
Coloring the $r_k$-separated sets

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Run-time comparison for coloring

<table>
<thead>
<tr>
<th>Level</th>
<th>Greedy Algorithm</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>360 ms</td>
<td>340 ms</td>
</tr>
<tr>
<td>6</td>
<td>2 sec</td>
<td>410 ms</td>
</tr>
<tr>
<td>8</td>
<td>26 min</td>
<td>5 sec</td>
</tr>
<tr>
<td>10</td>
<td>n/a</td>
<td>6 min</td>
</tr>
</tbody>
</table>
Embedding the Snowflaked Laakso Space

- Choose a sequence of scales $r_k$ (powers of a small parameter $\tau$).
- For each scale choose a maximal $r_k$-separated set of “grid points” in the metric space.
- Color the grid points at every level.
- Define the embedding based on the colorings of all the grid points.
- Scales $\leftrightarrow$ digits, and colors $\leftrightarrow$ coordinate directions (coordinate subspaces) of Euclidean space.
Defining the embedding

Fix a color $\xi$.

Assign each $\xi$-colored grid point a vector $v_J$ in the ball of radius $\tau^2$ in $\mathbb{R}^M$.

The function $F^\xi : X \rightarrow \mathbb{R}^m$ is a double weighted sum (weighted by level and proximity to grid points).
Choosing the vector $v_J$

Choose $v_J$ successively. For each choice, consider

- the weighted partial sum of previously chosen values in the annulus
- the weighted partial sum of previously chosen values in the ball together with the new $v_J$

The difference between these, for all choices of pairs of points, should be large.
Choosing the vector $\vec{v}_J$

Discretize. For fixed $x', y'$, at most one of the discrete vectors in the sphere doesn’t work as a choice of $v_J$.

For small $\tau$ and large $M$, there are more discrete vectors in the sphere than pairs $x', y'$ so one of the vectors in the sphere works for all $x'$ in the ball and $y'$ in the annulus.
Bounding the dimension of the target space

Given our $\tau = 1/64$, we find a sufficient dimension $M$ by determining the number of $(\tau^3 r_k)$-dense points in the ball $B_J$ and annulus $10B_J \setminus 2B_J$. 
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The discrete points can be taken to be the grid points 9 stages deeper than the $r_k$ level.

$$r_k = (1/4)^{6k} = 4^9(1/4)^{6k+9}$$

Hence, $\#$ pts in $B_J$ is

$$3(\# \text{ pts in level 9}) = 3 \left(2 + 4 \sum_{n=0}^{8} 6^n\right) = 6,046,623.$$
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Points in ball: $6 \times 10^6$
Points in annulus: $8 \times 10^{12}$
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Note that $\tau^2 / 7\tau^3 = 64/7$. Hence we need dimension of $\mathbb{R}^M$ so that there are $4.8 \times 10^{19}$ lattice points in the ball of radius 9.
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Use combinatorial formula for lattice points in $\ell_1$ balls: $M = 319$.

Conservative estimate of the final dimension is $2 \times 36 \times 319 = 22968$.

(Compare to a priori estimate of $2 \times 6^5 \times 2423174 \approx 3.8 \times 10^{10}$)
In Summary

Results so far

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- A non-greedy coloring algorithm.
- Dimension estimate of the target Euclidean space.
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- Find the vectors \( v_j \) for \( k = 1, 2 \). (Very computationally expensive. Hope to leverage symmetry.)
- Calculate embedded coordinates for \( k = 2 \).
- Investigate distortion of this embedding. Conservatively,
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  2^{-33} d(x, y)^{\alpha} \leq |F(x) - F(y)| \leq 2023 d(x, y)^{\alpha}
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  2^{-33} d(x, y)^\alpha \leq |F(x) - F(y)| \leq 2023 d(x, y)^\alpha
  \]
- Create/investigate visualizations using projections.
- Vary snowflaking constant and compare embeddings.
- Generalize methods/calculations to other fractals.