Combinatorial Atlas for Log-concave Inequalities

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joint with Igor Pak
What is log-concavity?

A sequence \( a_1, \ldots, a_n \in \mathbb{R}_{\geq 0} \) is log-concave if

\[
a_k^2 \geq a_{k+1} a_{k-1} \quad (1 \leq k < n).
\]

Equivalently,

\[
\log a_k \geq \frac{\log a_{k+1} + \log a_{k-1}}{2} \quad (1 \leq k < n).
\]
Example: binomial coefficients

\[ a_k = \binom{n}{k} \quad k = 0, 1, \ldots, n. \]

This sequence is log-concave because

\[
\frac{a_k^2}{a_{k+1} a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1} \binom{n}{k-1}} = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right),
\]

which is greater than 1.
Example: permutations with $k$ inversions

$$a_k = \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of $\pi$ is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \leq k \leq \binom{n}{2}} a_k q^k = [n]_q! = (1+q) \ldots (1+q \ldots + q^{n-1})$$

is a product of log-concave polynomials.

```
1 4 9 15 20 22 20 15 9 4 1
```
Log-concavity appears in different objects for different reasons.

Today we focus on reason for matroids.
Let $G = (V, E)$ be a graph.

A (spanning) forest $F = (V, E')$ with $E' \subseteq E$ is a subset of edges without cycles.
Log-concavity for forests

**Theorem (Huh ‘15)**

For every graph and \( k \geq 1 \),

\[
I_k^2 \geq I_{k+1} I_{k-1},
\]

where \( I_k \) is the number of forests with \( k \) edges.

Proof used Hodge theory from algebraic geometry.

In fact, stronger inequalities for more general objects are true.
Object: Matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set $X$ with collection of independent sets $\mathcal{I} \subseteq 2^X$.

Graphical matroids

- $X =$ edges of a graph $G$,
- $\mathcal{I} =$ forests in $G$.

Realizable matroids

- $X =$ finite set of vectors over field $\mathbb{F}$,
- $\mathcal{I} =$ sets of linearly independent vectors.
Matroids: Conditions

- $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.

If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.

Note: These are natural properties of sets of linearly independent vectors.
Mason’s Conjecture (1972)

For every matroid and $k \geq 1$,

(1) $I_k^2 \geq I_{k+1} I_{k-1};$

(2) $I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1};$

(3) $I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$

$I_k$ is number of ind. sets of size $k$, and $n = |X|$.

Note: (3) $\Rightarrow$ (2) $\Rightarrow$ (1).
Why \((1 + \frac{1}{k}) (1 + \frac{1}{n-k})\) ?

Mason (3) is equivalent to ultra/binomial log-concavity,

\[
\frac{I_k^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}}{\binom{n}{k+1}} \frac{I_{k-1}}{\binom{n}{k-1}}.
\]

Equality occurs if every subset with \(k + 1\) elements is independent.
Theorem (Adiprasito-Huh-Katz ‘18)

For every matroid and $k \geq 1$,

$$I_k^2 \geq I_{k+1} I_{k-1}.$$ 

Proof used combinatorial Hodge theory for matroids.
Solution to Mason (2)

Theorem (Huh-Schröter-Wang ‘18)
For every matroid and $k \geq 1$,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1}.$$ 

Proof used combinatorial Hodge theory for correlation inequality on matroids.
Solution to Mason (3)

**Theorem**

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh ‘20)

*For every matroid and \( k \geq 1 \),

\[
I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.
\]

Proof used theory of strong log-concave polynomials / Lorentzian polynomials.
Solution to Mason (3)

Theorem

\begin{equation}
I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{n-k} \right) I_{k+1} I_{k-1}.
\end{equation}

Theorem (Murai-Nagaoka-Yazawa ‘21)

Equality occurs if and only if every subset with \( k + 1 \) elements is independent.
Our contribution
Method: Combinatorial atlas

Results: Log-concave inequalities, and if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley’s poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids.
Method: Combinatorial atlas

Results: Log-concave inequalities, and if and only if conditions for equality

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- Interval greedoids.
Combinatorial atlas application:
Matroids
Warmup: graphical matroids refinement

**Corollary (C.-Pak)**

For graphical matroid of simple connected graph $G = (V, E)$, and $k = |V| - 2$,

$$(I_k)^2 \geq \frac{3}{2} \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1},$$

with equality if and only if $G$ is cycle graph.

Numerically better than Mason (3), because

$$\frac{3}{2} \geq 1 + \frac{1}{n-k} = 1 + \frac{1}{|E| - |V| + 2}$$

for $G$ that is not tree.
Comparison with Mason (3)

Our bound gives
\[
\frac{(I_k)^2}{I_{k+1} I_{k-1}} \geq \frac{3}{2} \quad \text{when} \quad |E| - |V| \to \infty,
\]

Meanwhile, Mason (3) bound only gives
\[
\frac{(I_k)^2}{I_{k+1} I_{k-1}} \geq 1 \quad \text{when} \quad |E| - |V| \to \infty.
\]

Our bound is better numerically and asymptotically.
Refinement for Mason (3)

**Theorem 1 (C.-Pak)**

For every matroid and \( k \geq 1 \),

\[
I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{\text{prl}_M(k - 1) - 1} \right) I_{k+1} I_{k-1}.
\]

This refines Mason (3),

\[
I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{n - k} \right) I_{k+1} I_{k-1},
\]

since

\[
\text{prl}_M(k - 1) \leq n - k + 1.
\]
Refinement for different matroids

- For all matroids,
  \[ I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{n-k} \right) I_{k+1} I_{k-1}. \]

- Graphical matroids and \( k = |V| - 2 \),
  \[ I_k^2 \geq \left( 1 + \frac{1}{k} \right) \frac{3}{2} I_{k+1} I_{k-1}. \]

- Realizable matroids over \( \mathbb{F}_q \),
  \[ I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{q^{m-k+1}-2} \right) I_{k+1} I_{k-1}. \]

- \((k, m, n)\)-Steiner system matroid,
  \[ I_k^2 \geq \left( 1 + \frac{1}{k} \right) \frac{n-k+1}{n-m} I_{k+1} I_{k-1}. \]
Refinement for Mason (3)

Theorem 2 (C.-Pak)

For every matroid and \( k \geq 1 \),

\[
I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{\text{prl}_\mathcal{M}(k-1)-1} \right) I_{k+1} I_{k-1}.
\]

This refines Mason (3),

\[
I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{n-k} \right) I_{k+1} I_{k-1},
\]

since

\[
\text{prl}_\mathcal{M}(k-1) \leq n - k + 1.
\]
Parallel classes of matroid $\mathcal{M}$

Loop is $x \in X$ such that $\{x\} \notin \mathcal{I}$.

Non-loops $x, y$ are parallel if $\{x, y\} \notin \mathcal{I}$.

Parallelship equiv. relation: $x \sim y$ if $\{x, y\} \notin \mathcal{I}$.

Parallel class $= \text{ equivalence class of } \sim$. 
Matroid contraction

Contraction of \( S \in \mathcal{I} \) is matroid \( \mathcal{M}_S \) with

\[
X_S = X \setminus S, \quad \mathcal{I}_S = \{ T \setminus S : S \subseteq T \}.
\]

\[
\text{prl}(S) := \text{number of parallel classes of } \mathcal{M}_S
\]
Parallel number

The $k$-parallel number is

$$\text{prl}_\mathcal{M}(k) := \max\{\text{prl}(S) \mid S \in \mathcal{I} \text{ with } |S| = k\}.$$
Theorem 3 (C.-Pak)

For every matroid and \( k \geq 1 \),

\[
I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{\text{prl}_M(k - 1) - 1} \right) I_{k+1} I_{k-1}.
\]

This refines Mason (3),

\[
I_k^2 \geq \left( 1 + \frac{1}{k} \right) \left( 1 + \frac{1}{n - k} \right) I_{k+1} I_{k-1},
\]

since

\[
\text{prl}_M(k - 1) \leq n - k + 1.
\]
When is equality achieved?

- When every \((k + 1)\)-subset is independent,
  \[\text{prl}_{\mathcal{M}}(k - 1) = n - k + 1.\]

- Graphical matroid when \(G\) is a cycle,
  \[\text{prl}_{\mathcal{M}}(k - 1) = 3.\]

- Realizable matroids of every \(m\)-vectors over \(\mathbb{F}_q\),
  \[\text{prl}_{\mathcal{M}}(k - 1) = q^{m-k+1} - 1.\]

- \((k, m, n)\)-Steiner system matroid,
  \[\text{prl}_{\mathcal{M}}(k - 1) = \frac{n - k + 1}{m - k + 1}.\]
Equality conditions

Theorem 4 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k^2 = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}_M(k - 1) - 1}\right) I_{k+1} I_{k-1}$$

if and only if

for every $S \in \mathcal{I}$ with $|S| = k - 1$,

- $M_S$ has $\text{prl}_M(k - 1)$ parallel classes; and
- Every parallel class of $M_S$ has same size.
Combinatorial atlas: the method
Combinatorial atlas

**Input**: Acyclic digraph $\mathcal{A}$, where each vertex $v$ is associated with
- Symmetric matrix $M$ with nonnegative entries;
- Vector $g, h$ with nonnegative entries.

**Goal**: Show every $M$ has hyperbolic inequality.

**Method**: Verify two conditions:
- Inheritance conditions
- Subdivergence conditions
Atlas: example
Atlas: example (zoomed in)
Atlas example: matroid (simplified)

For matroid with $X = \{a, b, c\}$, the atlas for $k = 2$ is
Atlas example: matroid (simplified)

The matrix for the top vertex is

\[ M_{a,b} = (k + 1)! \times \text{number of independent sets of size } k + 1 \text{ containing } a, b \]

\[ M_{a,*} = k! \times \text{number of independent sets of size } k \text{ containing } a \]

\[ M_{*,*} = (k - 1)! \times \text{number of independent sets of size } k - 1 \]
Combinatorial atlas

**Input:** Acyclic digraph $\mathcal{A}$, where each vertex $v$ is associated with

- Symmetric matrix $M$ with nonnegative entries;
- Vector $g, h$ with nonnegative entries.

**Goal:** Show every $M$ has hyperbolic inequality.
Hyperbolic inequality

$M$ has hyperbolic inequality property if

$$\langle x, My \rangle^2 \geq \langle x, Mx \rangle \langle y, My \rangle,$$

for every $x \in \mathbb{R}^r$, $y \in \mathbb{R}_{\geq 0}^r$.

This condition is equivalent to

$M$ has at most one positive eigenvalue.

Note: Already known to be important in Lorentzian polynomials and Bochner’s method proof of Aleksandrov-Fenchel inequality.
How to get log-concave inequalities?

Assume $a_{k-1}, a_k, a_{k+1}$ can be computed by

$$a_k = \langle g, Mh \rangle, \quad a_{k+1} = \langle g, Mg \rangle, \quad a_{k-1} = \langle h, Mh \rangle,$$

for $M, g, h$ from a top vertex of the atlas.

\[
\langle g, Mh \rangle^2 \geq \langle g, Mg \rangle \langle h, Mh \rangle \quad \text{(hyperbolic ineq.)}
\]

then implies

$$a_k^2 \geq a_{k+1}a_{k-1} \quad \text{(log-concave ineq.)}$$
Combinatorial atlas

**Input:** Acyclic digraph $\mathcal{A}$, where each vertex $v$ is associated with

- Symmetric matrix $M$ with nonnegative entries;
- Vector $g, h$ with nonnegative entries.

**Goal:** Show every $M$ has hyperbolic inequality.

**Method:** Verify three conditions:

- Irreducibility condition;
- Inheritance condition;
- Subdivergence condition.
Irreducibility condition

- Matrix $\mathbf{M}$ associated to $\nu$ is irreducible when restricted to its support;
- Vector $\mathbf{h}$ is associated to $\nu$ is a positive vector.

For matroids, this means that the base exchange graph is connected.
This is a consequence of the exchange property.
Inheritance condition

Edge $e = (v, v_i)$ of $v$ is associated with linear map $T_i : \mathbb{R}^r \rightarrow \mathbb{R}^r$ such that, for every $x \in \mathbb{R}^r$,

$$i\text{-th coordinate of } Mx = \langle T_i x, M_i T_i h \rangle,$$

where $M$ and $h$ are associated to $v$, and $M_i$ is associated to $v_i$.

For matroids with $X = \{e_1, \ldots, e_n\}$, this means

$$k \times \text{number of independent } k\text{-sets}$$

$$= \sum_{i=1}^{n} \text{number of independent } k\text{-sets containing } e_i.$$
Subdivergence condition

For every $x \in \mathbb{R}^r$,

\[
\sum_{i=1}^{r} h_i \langle T_i x, M_i T_i x \rangle \geq \langle x, Mx \rangle,
\]

where $h_i = i$-th coordinate of $h$.

Note: Equality occurs for Lorentzian polynomials and for matroids.

For matroids, this is consequence of hereditary property.
Bottom-to-top principle for hyperbolic inequalities

Proposition
Assume irreducibility, inheritance, subdvergence. If every child vertex has hyperbolic inequality property, then so does the parent vertex.

Bottom-to-top principle reduces Goal to checking hyperbolic inequality only for sink vertices.
Bottom-to-top principle
Bottom-to-top principle
Bottom-to-top principle
Bottom-to-top principle
How about equalities?
Combinatorial atlas equality

Input:
- An atlas $\mathcal{A}$ satisfying irreducibility, inheritance, subdivergence conditions.

Goal: Show “every” $M$ has hyperbolic equality,

$$\langle g, Mh \rangle^2 = \langle g, Mg \rangle \langle h, Mh \rangle.$$
Proposition

If parent vertex has hyperbolic equality property, then so does children vertices.

Top-to-bottom principle expands hyperbolic equality to sink vertices, and gives combinatorial characterizations.
Top-to-bottom principle
Top-to-bottom principle
Top-to-bottom principle
Top-to-bottom principle
Moral of the story

**Problem:** Log-concave inequalities and equalities.

**Strategy:**
- Build a combinatorial atlas;
- Verify the required conditions;
- Use hyperbolic inequality property to derive log-concave inequalities;
- Use hyperbolic equality property to derive log-concave equalities.
Other applications

Full version: 2110.10740 (71 pages)
Expository version: 2203.01533 (28 pages)

Results: Log-concave inequalities and equalities for

- Matroids (refined);
- Discrete polymatroids;
- Morphism of matroids (refined) (conjecture on equality conditions is resolved);
- Stanley’s poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids.
THANK YOU!

            www.arxiv.org/abs/2203.01533

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Negative dependence for forests

**Conjecture (Kahn '00, Grimmett-Winkler '04)**

Let $G$ be a graph, let $e, f$ be distinct edges of $G$. Then

$$P[e, f \in F] \leq P[e \in F] P[f \in F],$$

where $F$ is uniform random forest of $G$.

- Known with extra factor of 2 in RHS by Lorentzian polynomials
- For matroids, the conjectured factor is $\frac{8}{7}$. 
Combinatorial atlas application:
Stanley’s poset inequality
A poset $P$ is a set $X$ with a partial order $\prec$ on $X$. 
Linear extension

A linear extension $L$ is a complete order of $\prec$.

We write $L(x) = k$ if $x$ is $k$-th smallest in $L$. 
Stanley’s inequality

Fix \( z \in P \).

\( N_k \) is number of linear extensions with \( L(z) = k \).

**Theorem (Stanley ‘81)**

*For every poset and \( k \geq 1 \),*

\[
N_k^2 \geq N_{k+1} N_{k-1}.
\]

Proof used **Aleksandrov-Fenchel inequality** for mixed volumes.
When is equality achieved?

**Theorem (Shenfeld-van Handel)**

Suppose $N_k > 0$. Then

$$N_k^2 = N_{k+1} N_{k-1}$$

if and only if

$$N_k = N_{k+1} = N_{k-1}.$$

Proof used classifications of extremals of Aleksandrov-Fenchel inequality for convex polytopes.
Our contribution

Open Problem (Folklore)

Give a combinatorial proof to Stanley’s inequality.

Answer (C.–Pak)

We give new combinatorial proof for Stanley’s ineq. and extend to weighted version.
Order-reversing weight

A weight $w : X \to \mathbb{R}_{>0}$ is order-reversing if

$$w(x) \geq w(y) \quad \text{whenever} \quad x \prec y.$$ 

Weight of linear extension $L$ is

$$w(L) := \prod_{L(x) < L(z)} w(x).$$

$$w(L) = w(a)$$

$$w(L) = w(a)w(b)$$
Weighted Stanley’s inequality

Fix $z \in P$.

$N_{w,k}$ is $w$-weight of linear extensions with $L(z) = k$.

**Theorem 5 (C. Pak)**

*For every poset and $k \geq 1$,*

$$N_{w,k}^2 \geq N_{w,k+1} N_{w,k-1}.$$
When is equality achieved?

Theorem 6 (C.-Pak)

Suppose $N_{w,k} > 0$. Then

$$N_{w,k}^2 = N_{w,k+1}N_{w,k-1}$$

if and only if

for every linear extension $L$ with $L(z) = k$,

$$w(L^{-1}(k + 1)) = w(L^{-1}(k - 1)) =: s,$$

and

$$\frac{N_{w,k}}{s^k} = \frac{N_{w,k+1}}{s^{k+1}} = \frac{N_{w,k-1}}{s^{k-1}}.$$
Combinatorial atlas application: Poset antimatroids
Feasible words of a poset

A word $\alpha \in X^*$ is feasible if no repeating elements, and $y$ occurs in $\alpha$ and $x \prec y$ $\Rightarrow$ $x$ occurs in $\alpha$ before $y$.

Feasible: $\emptyset$, $a$, $ab$, $ac$, $abc$, $acb$, $abcd$, $acbd$.

Not feasible: $aa$, $bc$, $ba$. 

\[
\begin{array}{cccc}
\text{d} & \text{c} \\
\text{b} & \text{a}
\end{array}
\]
Chain weight

For \( x \in P \), chain weight is \( \omega(x) = \) number of maximal chains that starts with \( x \).

\[
\begin{align*}
\omega(a) &= 2 \\
\omega(b) &= 1 \\
\omega(c) &= 1 \\
\omega(d) &= 1
\end{align*}
\]

Weight of word \( \alpha \) is \( \omega(\alpha) := \omega(\alpha_1) \ldots \omega(\alpha_\ell) \).
Log-concave inequality for poset antimatroids

$F_{\omega,k}$ is sum of $\omega$-weight of feasible words of length $k$.

**Theorem 7 (C.-Pak)**

For every poset and $k \geq 1$,

$$F_{\omega,k}^2 \geq F_{\omega,k+1} F_{\omega,k-1}.$$
When is equality achieved?

Theorem 8 (C.-Pak)

Equality occurs for \( k = 1, \ldots, \text{height}(P) - 1 \)

if and only if

Hasse diagram of \( P \) is a forest where every leaf is of the same level.