Sorting probability for Young diagrams

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joint with Igor Pak and Greta Panova
Partially ordered set

A poset $P$ is a set $X$ with a partial order $\preceq$ on $X$. 
A linear extension $L$ is a complete order of $\preceq$.

We write $e(P)$ for number of linear extensions of $P$. 
How many steps needed to complete a partial order?
How many steps needed to complete a partial order?

We first compare $c$ and $d$, and get $c \preceq d$. 
How many steps needed to complete a partial order?

We then compare $d$ and $e$, and get $d \preceq e$. 
How many steps needed to complete a partial order?

We continue with $b$ and $e$, and get $e \preceq b$. 
How many steps needed to complete a partial order?

Completing the partial order took 3 steps.
Strategy to complete the partial order

At each step, compare $x$ and $y$ that satisfies

$$\frac{1}{2} - c \leq P\left[x \preceq y\right] \leq \frac{1}{2} + c,$$

where $P$ is uniform on linear extensions of $P$.

Runtime is $\Theta(\log e(P))$ steps.
Conjecture (Kislitsyn ’68, Fredman ’75, Linial ’84)

For every finite poset that is not completely ordered, there exists \( x, y \):

\[
\frac{1}{3} \leq P[x \preceq y] \leq \frac{2}{3}.
\]

(Brightwell-Felsner-Trotter ’95)

“This problem remains one of the most intriguing problems in the combinatorial theory of posets.”
Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper, lower bound are achieved by this poset:

\[
\begin{align*}
\text{P}[x \preceq y] &= \frac{1}{3}; \\
\text{P}[y \preceq x] &= \frac{2}{3}.
\end{align*}
\]
Theorem (Kahn-Saks ’84)

For every finite poset, there always exists $x, y$:

\[
\frac{3}{11} \leq P[x \preceq y] \leq \frac{8}{11},
\]

roughly between 0.273 and 0.727.

Proof is by applying mixed-volume inequalities to order polytopes.
What is known so far

**Theorem (Brightwell-Felsner-Trotter ’95)**

For every finite poset, there always exists $x, y$:

$$\frac{5 - \sqrt{5}}{10} \leq P[x \preceq y] \leq \frac{5 + \sqrt{5}}{10},$$

roughly between 0.276 and 0.724.

This bound cannot be improved for infinite posets.
Young diagrams

Elements of $P_{\lambda}$ are **cells** of Young diagram of shape $\lambda$.

$x \preceq y$ if $y$ lies to the Southeast of $x$.

Young diagram of shape $\lambda = (4, 3, 1)$

We write $n$ for **number of cells** of Young diagram.
Young diagrams

Linear extensions of $P_\lambda$ correspond to standard Young tableau of the Young diagram.

Linear extensions are counted by hook-length formulas.
What is known for Young diagrams

**Theorem 1 (Olson–Sagan ’18)**

For Young diagrams, there always exists $x, y$:

$$\frac{1}{3} \leq P[x \preceq y] \leq \frac{2}{3}.$$
What is known for Young diagrams

**Theorem 1 (Olson–Sagan ’18)**

*For Young diagrams, there always exists* \( x, y : \)

\[
\frac{1}{3} \leq P[x \preceq y] \leq \frac{2}{3}.
\]

We sketch an alternative proof for Young diagrams using **Naruse hook-length formulas**.
Hook-length formulas

Number of standard Young tableau of shape $\lambda$ is

$$f^\lambda := \frac{n!}{\prod_{x \in \lambda} h^\lambda(x)}.$$
Skew Young diagram of shape $\lambda/\mu$,
$\lambda = (5, 3, 3, 1)$ and $\mu = (2, 1)$.

We write $n$ for number of cells in $\lambda$, and $m$ for number of cells in $\mu$. 
Excited diagrams

Black boxes can move on SouthEast direction.
Naruse hook-length formulas

**Theorem (Naruse ’14, Morales-Pak-Panova ’17)**

Number of skew Young tableau of shape $\lambda/\mu$ is

$$f^{\lambda/\mu} := f^{\lambda} \frac{(n - m)!}{n!} \sum_{\text{excited diagrams } B} \prod_{x \in B} h^{\lambda}(x).$$
The number of SYT of shape $\lambda/\mu$ is equal to

$$2970 \cdot \frac{9!}{12!} \left( 7 \cdot 6 \cdot 5 + 7 \cdot 5 \cdot 2 + 7 \cdot 2 \cdot 3 + 7 \cdot 6 \cdot 3 + 4 \cdot 2 \cdot 3 \right)$$

$$= 1062.$$
The jump probabilities are

\[ p_i := P\left[y_i \preceq x \preceq y_{i+1}\right] \]
Proof of Theorem Olson–Sagan

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Proof of Theorem Olson–Sagan

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\[ p_i := \Pr[y_i \preceq x \preceq y_{i+1}] \]
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Proof of Theorem Olson–Sagan

The jump probabilities are

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Linial-type argument

Supppose that $p_1, p_2, \ldots, p_\ell$ are all $< \frac{1}{3}$.

Look at when the probability exceeds $\frac{1}{3}$. Then

$$\frac{1}{3} \leq P\left[ x \preceq y_{i+1} \right] \leq \frac{2}{3}.$$
Proof of \( p_1 < \frac{1}{3} \)

Suppose to the contrary that \( p_1 \geq \frac{1}{3} \). Then

- If \( \frac{1}{3} \leq p_1 \leq \frac{2}{3} \), then
  \[
  \frac{1}{3} \leq p_1 = P[x \preceq y_2] \leq \frac{2}{3}.
  \]

- If \( p_1 > \frac{2}{3} \), then conjugate to get \( p_1 < \frac{1}{3} \).
Skew diagrams enter the scene

It suffices to show $p_1 \geq p_2 \geq \ldots \geq p_\ell$.

\[
p_1 = P[y_1 \preceq x \preceq y_2] = \frac{\text{\# of SYTs of } f^\lambda}{f^\lambda}
\]

\[
p_2 = P[y_2 \preceq x \preceq y_3] = \frac{\text{\# of SYTs of } f^\lambda}{f^\lambda}
\]
Skew diagrams enter the scene

It suffices to show \( p_1 \geq p_2 \geq \ldots \geq p_\ell \).

\[
p_1 = \mathbb{P}(y_1 \preceq x \preceq y_2) = \frac{\# \text{ of SYTs of } f^\lambda}{f^\lambda}
\]

\[
p_2 = \mathbb{P}(y_2 \preceq x \preceq y_3) = \frac{\# \text{ of SYTs of } f^\lambda}{f^\lambda}
\]

We can now use NHLF.
Proof of $p_1 \geq p_2$

\[
p_1 = \frac{(10! \cdot 7 \cdot 6 + 10! \cdot 7 \cdot 2 + 10! \cdot 4 \cdot 2)}{12!} = \frac{9!}{12!} \cdot 640.
\]

\[
p_2 = \frac{(9! \cdot 7 \cdot 6 \cdot 8 + 9! \cdot 7 \cdot 2 \cdot 8 + 9! \cdot 4 \cdot 2 \cdot 3)}{12!} = \frac{9!}{12!} \cdot 472.
\]
Thus we complete the proof of this theorem.

**Theorem (Olson–Sagan ’18)**

*There always exists* $x, y$:

\[
\frac{1}{3} \leq P[x \lessdot y] \leq \frac{2}{3},
\]

*for poset $P_\lambda$ of Young diagram of shape $\lambda$.***
Comparison probability for this Young diagram is

\[
\Pr[ x \preceq y ] = \frac{16}{33} \approx 0.4848,
\]

which is closer to \( \frac{1}{2} \) than \( \frac{1}{3}, \frac{2}{3} \).
What we will do next

Previously, we want to find $x, y$:

$$\frac{1}{3} \leq P[x \preceq y] \leq \frac{2}{3},$$

Now, we want to find $x, y$:

$$\frac{1}{2} - \delta \leq P[x \preceq y] \leq \frac{1}{2} + \delta,$$
Sorting probability of a poset $P$ is

$$\delta(P) := \min_{\text{distinct } x, y} \left| P[x \prec y] - P[y \prec x] \right|.$$  

In particular, there exists $x, y$:

$$\frac{1}{2} - \frac{\delta(P)}{2} \leq P[x \preceq y] \leq \frac{1}{2} + \frac{\delta(P)}{2}.$$
Kahn–Saks Conjecture

Conjecture (Kahn-Saks ’84)

For every finite poset,

\[ \delta(P) \rightarrow 0 \quad \text{as} \quad \text{width}(P) \rightarrow \infty. \]

Here \( \text{width}(P) \) is the largest size of anti-chains in \( P \).

Komlós ’90 proved such a result for posets with \( \Omega\left(\frac{n}{\log \log \log n}\right) \) minimal elements.
Our results
First result

Theorem (C.-Pak-Panova ’20+)

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset $P_\lambda$ of Young diagram of $\lambda$,

$$\delta(P_\lambda) \leq \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$. 

\[ 
\begin{array}{cccccccccc}
\text{d} & & & & & & & & & \\
\varepsilon n & & & & & & & & & \\
\end{array} 
\]
Where is the improvement?

Before: $x$ is 2nd element in 1st row, $y$ is in 1st column.

Now: $x$ is middle element in 1st row, $y$ is in 2nd row.
Where is the improvement?

Before: \( x \) is 2nd element in 1st row, \( y \) is in 1st column.

Now: \( x \) is middle element in 1st row, \( y \) is in 2nd row.
Where is the improvement?

Before: $x$ is 2nd element in 1st row, $y$ is in 1st column.

\[
\begin{array}{cccccccc}
 & & & & & & & x \\
 & & & & & & y & \\
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\end{array}
\]

Now: $x$ is middle element in 1st row, $y$ is in 2nd row.

\[
\begin{array}{cccccccc}
 & & & & & & & x \\
 & & & & & & y & \\
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\end{array}
\]
Where is the improvement?

Before: \( x \) is 2nd element in 1st row, \( y \) is in 1st column.

Now: \( x \) is middle element in 1st row, \( y \) is in 2nd row.
Where is the improvement?

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Now: $x$ is **middle** element in 1st row, $y$ is in 2nd row.

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Sketch of proof

After reductions using Hoeffding’s inequality,

\[ \delta(P_\lambda) \leq \sum_{\mu} \frac{\text{SYTs of } f_\lambda}{\mu} \]

with \( \mu \approx \left( \frac{\lambda_1}{2} \pm \sqrt{n}, \ldots, \frac{\lambda_d}{2} \pm \sqrt{n} \right). \)

Right side is then upper-bounded via NHLF.
Theorem (C.-Pak-Panova ’20+)

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset $P_\lambda$ of Young diagram of $\lambda$,

$$\delta(P_\lambda) \leq \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$.

Next: better bound for Catalan posets.
Catalan posets, $\lambda = \left( \frac{n}{2}, \frac{n}{2} \right)$

Young diagram is rectangle with 2 rows and $n$ cells.
Second result

**Theorem (C.-Pak-Panova '21)**

For *Catalan posets with n cells*,

\[ \delta(P_\lambda) \leq C n^{-\frac{5}{4}}, \]

for some \( C > 0 \).
How good is this bound?

Open Problem

Show that

$$\limsup_{n \to \infty} \frac{\log \delta(P)_{\lambda}}{n} = -\frac{5}{4}; \quad \liminf_{n \to \infty} \frac{\log \delta(P)_{\lambda}}{n} < -\frac{5}{4}.$$
Where is the improvement?

Before: $x$ is fixed at midpoint, only $y$ is optimized.

Now: Optimize $y = y(x)$ for each $x$, then optimize $x$.

For each $x$, $y(x)$ is the element that minimizes

$$
\delta(x, y(x)) := \left| P [x \prec y(x)] - P [y(x) \prec x] \right|.
$$
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Before: \( x \) is fixed at midpoint, only \( y \) is optimized.

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\]

Now: Optimize \( y = y(x) \) for each \( x \), then optimize \( x \).

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& & & & & \\
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& & x & & & \\
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& y(x) & & & & \\
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\end{array}
\]

Now: Optimize $y = y(x)$ for each $x$, then optimize $x$.

\[
\begin{array}{|c|c|c|c|c|c|}
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\delta(x, y(x)) := \left| \Pr [x \prec y(x)] - \Pr [y(x) \prec x] \right|.
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\end{array}
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\end{array}
\]

For each \( x \), \( y(x) \) is the element that minimizes

\[
\delta(x, y(x)) := | P [x \preceq y(x)] - P [y(x) \preceq x] |
\]
Where is the improvement?

Before: $x$ is fixed at midpoint, only $y$ is optimized.

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Now: **Optimize** $y = y(x)$ for each $x$, then **optimize** $x$.

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For each $x$, $y(x)$ is the element that minimizes

$$
\delta(x, y(x)) := \left| P \left[ x \prec y(x) \right] - P \left[ y(x) \prec x \right] \right|.
$$
Location of the optimizer $y(x)$ for $n = 2000$

For each $x$, $y(x)$ is the element that minimizes

$$\delta(x, y(x)) := \left| P [x \prec y(x)] - P [y(x) \prec x] \right|.$$
Sorting probability $\delta(P)$ for $n = 2000$

\[ \delta(x, y(x)) := \left| P[x \prec y(x)] - P[y(x) \prec x] \right|. \]
Back to second result

**Theorem (C.-Pak-Panova ’21)**

For Catalan posets with $n$ cells,

$$\delta(P_\lambda) \leq C n^{-\frac{5}{4}},$$

for some $C > 0$.

**Important:** Estimates are not done by NHLF, but by direct computation.

Better upper bound for general Young diagrams remain open.
What is next?

**Theorem (C.-Pak-Panova ’20+)**

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset $P_\lambda$ of Young diagram of $\lambda$, there exists $x, y$:

$$\delta(P_\lambda) \to 0 \quad \text{as} \quad n \to \infty.$$

**Open Problem**

Prove same result for other families of posets, e.g., $k$-dimensional Young diagrams and periodic posets.
Webpage: http://math.ucla.edu/~sweehong/
THANK YOU!

Webpage: http://math.ucla.edu/~sweehong/