Combinatorial Atlas for Log-concave Inequalities

Swee Hong Chan (UCLA)

joint with Igor Pak
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \geq a_{k+1} a_{k-1} \quad (1 \leq k < n).$$

Equivalently,

$$\log a_k \geq \frac{\log a_{k+1} + \log a_{k-1}}{2} \quad (1 \leq k < n).$$

1 4 9 15 20 22 20 15 9 4 1
Example: binomial coefficients

\[ a_k = \binom{n}{k} \quad k = 0, 1, \ldots, n. \]

This sequence is log-concave because

\[
\frac{a_k^2}{a_{k+1}a_{k-1}} = \frac{(n)^2}{\binom{n}{k+1}\binom{n}{k-1}} = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),
\]

which is greater than 1.
Example: permutations with $k$ inversions

$$a_k = \text{number of } \pi \in S_n \text{ with } k \text{ inversions,}$$

where inversion of $\pi$ is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \leq k \leq \binom{n}{2}} a_k x^k = (1 + x) \ldots (1 + x + \ldots + x^{n-1})$$

is a product of log-concave polynomials.
Log-concavity appears in many objects:

algebras, matroids, mixed volumes, measures, posets, random walks.
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Today we focus on matroids and posets.
Matroids

Matroid $\mathcal{M}$ is ground set $X$ with collection of independent sets $\mathcal{I} \subseteq 2^X$,

- $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.

- If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$. 
Examples: Matroids

Graphical matroids
- $X = \text{edges of a graph } G$,
- $\mathcal{I} = \text{forests in } G$.

Realizable matroids
- $X = \text{finite set of vectors over field } \mathbb{F}$,
- $\mathcal{I} = \text{sets of linearly independent vectors.}$
Mason’s Conjecture (1972)

For every matroid and $k \geq 1$,

1. $I_k^2 \geq I_{k+1} I_{k-1}$;

2. $I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1}$;

3. $I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}$.

$I_k$ is number of ind. sets of size $k$, and $n = |X|$.
Why \((1 + \frac{1}{k})(1 + \frac{1}{n-k})\)?

Mason (3) is equivalent to **ultra log-concavity**,

\[
\frac{I_k^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}}{\binom{n}{k+1}} \frac{I_{k-1}}{\binom{n}{k-1}}.
\]

Equality occurs **if** every \((k + 1)\)-subset is independent.
Solution to Mason (1)

Theorem (Adiprasito-Huh-Katz ‘18)

For every matroid and \( k \geq 1 \),

\[
I_k^2 \geq I_{k+1} I_{k-1}.
\]

Proof used combinatorial Hodge theory for matroids.
Solution to Mason (2)

Theorem (Huh-Schröter-Wang ‘18)
For every matroid and $k \geq 1$, 

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1}.$$ 

Proof used combinatorial Hodge theory for correlation bound on matroids.
Solution to Mason (3)

Theorem (Anari-Liu-Gharan-Vinzant, Brändén-Huh ‘20)
For every matroid and $k \geq 1$,

\[ I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}. \]

Proof used theory of strong log-concave polynomials / Lorentzian polynomials.
**Theorem** (Anari-Liu-Gharan-Vinzant, Brändén-Huh ‘20)

*For every matroid and $k \geq 1$,*

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$$ 

**Theorem** (Murai-Nagaoka-Yazawa ‘21)

*Equality occurs if and only if every $(k+1)$-subset is independent.*
Our contribution
Method: Combinatorial atlas

Results: Log-concave inequalities, and if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley’s poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex).
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Matroids
Corollary (C.-Pak)

For graphical matroid of simple connected graph $G = (V, E)$ that is not tree, and $k = |V| - 2$,

$$(I_k)^2 \geq \frac{3}{2} \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1},$$

with equality if and only if $G$ is cycle graph.

Numerically better than Mason (3), because

$$\frac{3}{2} \geq 1 + \frac{1}{n - k} = 1 + \frac{1}{|E| - |V| + 2}.$$
Comparison with Mason (3)

Our bound gives

\[ \frac{(I_k)^2}{I_{k+1} I_{k-1}} \geq \frac{3}{2} \quad \text{when} \quad |E| - |V| \to \infty, \]

Meanwhile, Mason (3) bound only gives

\[ \frac{(I_k)^2}{I_{k+1} I_{k-1}} \geq 1 \quad \text{when} \quad |E| - |V| \to \infty. \]

Our bound is better numerically and asymptotically.
Parallel classes of matroid $\mathcal{M}$

Loop is $x \in X$ such that $\{x\} \notin \mathcal{I}$.

Non-loops $x, y$ are parallel if $\{x, y\} \notin \mathcal{I}$.

Parallelship equiv. relation: $x \sim y$ if $\{x, y\} \notin \mathcal{I}$.

Parallel class $=$ equivalence class of $\sim$. 
Matroid contraction

Contraction of $S \in \mathcal{I}$ is matroid $\mathcal{M}_S$ with

$$X_S = X \setminus S, \quad \mathcal{I}_S = \{ T \setminus S : S \subseteq T \}.$$ 

$$\operatorname{prl}(S) := \text{number of parallel classes of } \mathcal{M}_S$$
Parallel number

The $k$-parallel number is

$$\text{prl}(k) := \max\{\text{prl}(S) \mid S \in \mathcal{I} \text{ with } |S| = k\}.$$
For every matroid and $k \geq 1$,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}(k - 1) + 1}\right) I_{k+1} I_{k-1}.$$ 

This refines Mason (3),

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n - k}\right) I_{k+1} I_{k-1},$$

since

$$\text{prl}(k - 1) \leq n - k + 1.$$
When is equality achieved?

- When every \((k + 1)\)-subset is independent,
  \[ \text{prl}(k - 1) = n - k + 1. \]

- Graphical matroid when \(G\) is a cycle,
  \[ \text{prl}(k - 1) = 3. \]

- Full realizable matroids over finite field \(\mathbb{F}_q\),
  \[ \text{prl}(k - 1) = \frac{n}{q^{k-1}} - 1. \]

- \((k, m, n)\)-Steiner system matroid,
  \[ \text{prl}(k - 1) = \frac{n - k + 1}{m - k + 1}. \]
Equality conditions

**Theorem 2 (C.-Pak)**

For every matroid and $k \geq 1$,

\[
I_k^2 = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}(k-1) + 1}\right) I_{k+1} I_{k-1}
\]

if and only if

for every $S \in \mathcal{I}$ with $|S| = k - 1$,

- $S$ has $\text{prl}(k-1)$ parallel classes; and
- Every parallel class of $S$ has same size.
Stanley’s poset inequality
A poset $P$ is a set $X$ with a partial order $\prec$ on $X$. 
Linear extension

A linear extension $L$ is a complete order of $\prec$.

We write $L(x) = k$ if $x$ is $k$-th smallest in $L$. 
Stanley’s inequality

Fix $z \in P$.

$N_k$ is number of linear extensions with $L(z) = k$.

**Theorem (Stanley ‘81)**

For every poset and $k \geq 1$,

$$N_k^2 \geq N_{k+1} \cdot N_{k-1}.$$

Proof used **Aleksandrov-Fenchel inequality** for mixed volumes.
When is equality achieved?

**Theorem (Shenfeld-van Handel)**

Suppose $N_k > 0$. Then

$$N_k^2 = N_{k+1} N_{k-1}$$

if and only if

$$N_k = N_{k+1} = N_{k-1}.$$  

Proof used classifications of extremals of Aleksandrov-Fenchel inequality for convex polytopes.
Kahn–Saks inequality

Fix $x, y \in P$.

$E_k$ is number of lin. exts. with $L(y) - L(x) = k$.

**Theorem (Kahn–Saks ‘84)**

For every poset and $k \geq 1$,

$$E_k^2 \geq E_{k+1} E_{k-1}.$$  

Crucial component in the proof of first known bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture.
Kahn–Saks equality

**Conjecture**

Suppose $E_k > 0$. Then

$$E_k^2 = E_{k+1} E_{k-1}$$

if and only if

$$E_k = E_{k+1} = E_{k-1}.$$ 

Verified by C.–Pak–Panova for width two posets.
Our contribution

We give new *combinatorial proof* for Stanley’s ineq. and extend to *weighted version*. 
Order-reversing weight

A weight $w : X \to \mathbb{R}_{>0}$ is order-reversing if

$$w(x) \geq w(y) \quad \text{whenever} \quad x \prec y.$$

Weight of linear extension $L$ is

$$w(L) := \prod_{L(x) < L(z)} w(x).$$
Weighted Stanley’s inequality

Fix $z \in P$.

$N_{w,k}$ is $w$-weight of linear extensions with $L(z) = k$.

**Theorem 3 (C. Pak)**

*For every poset and $k \geq 1$,*

$$N_{w,k}^2 \geq N_{w,k+1} N_{w,k-1}.$$
When is equality achieved?

Theorem 4 (C.-Pak)

Suppose $N_{w,k} > 0$. Then

$$N_{w,k}^2 = N_{w,k+1}N_{w,k-1}$$

if and only if

for every linear extension $L$ with $L(z) = k$,

$$w(L^{-1}(k + 1)) = w(L^{-1}(k - 1)) =: s,$$

and

$$\frac{N_{w,k}}{s^k} = \frac{N_{w,k+1}}{s^{k+1}} = \frac{N_{w,k-1}}{s^{k-1}}.$$
Poset antimatroids
Feasible words of a poset

A word $\alpha \in X^*$ is feasible if no repeating elements, and $y$ occurs in $\alpha$ and $x \prec y \implies x$ occurs in $\alpha$ before $y$.

Feasible: $\emptyset$, $a$, $ab$, $ac$, $abc$, $acb$, $abcd$.
Not feasible: $aa$, $bc$, $ba$. 
Chain weight

For \( x \in P \), chain weight is
\[ \omega(x) = \text{number of maximal chains that starts with } x. \]

\[
\begin{align*}
\omega(a) &= 2 \\
\omega(b) &= 1 \\
\omega(c) &= 1 \\
\omega(d) &= 1
\end{align*}
\]

Weight of word \( \alpha \) is \( \omega(\alpha) := \omega(\alpha_1) \ldots \omega(\alpha_\ell) \).
Log-concave inequality for poset antimatroids

Theorem 5 (C.-Pak)

For every poset and $k \geq 1$,

$$F_{\omega,k}^2 \geq F_{\omega,k+1} F_{\omega,k-1}.$$
When is equality achieved?

**Theorem 6 (C.-Pak)**

Equality occurs for \( k = 1, \ldots, \text{height}(P) - 1 \)

if and only if

Hasse diagram of \( P \) is a forest where every leaf is of the same level.
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Combinatorial atlas
The strategy

**Input:** Acyclic digraph $\mathcal{A}$, where each vertex $v$ is associated with

- $r \times r$ nonnegative symmetric matrix $M$;
- nonnegative $r$-vector $h$. 
Combinatorial atlas: example

\[ \begin{array}{c}
 a & b \\
 c & d & e & f \\
 g & h & i & j & k & \ell
\end{array} \]
Combinatorial atlas: example (zoomed in)
Combinatorial atlas: Example Stanley’s Inequality

\[
\text{Poset} = \begin{array}{cccc}
  & x & \circ & \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

- \( x \rightarrow y \rightarrow z \)
- \( y \rightarrow x \rightarrow z \)
- \( x \rightarrow z \rightarrow y \)
- \( z \rightarrow x \rightarrow y \)
- \( z \rightarrow y \rightarrow x \)
The strategy

**Input:** Acyclic digraph $\mathcal{A}$, where each vertex $v$ is associated with
- $r \times r$ nonnegative symmetric matrix $M$;
- A nonnegative $r$-vector $h$.

**Goal:** Show every $M$ has hyperbolic inequality.
Hyperbolic inequality

\( M \) has hyperbolic inequality property if

\[ \langle x, My \rangle^2 \geq \langle x, Mx \rangle \langle y, My \rangle, \]

for every \( x \in \mathbb{R}^r, y \in \mathbb{R}^r_{\geq 0} \).

**Note:** This property already known to be important in Lorentzian polynomials and Bochner’s method proof of Aleksandrov-Fenchel inequality.
How to get log-concave inequalities?

Assume $a_{k-1}, a_k, a_{k+1}$ can be computed by

$$a_k = \langle g, Mh \rangle, \quad a_{k+1} = \langle g, Mg \rangle, \quad a_{k-1} = \langle h, Mh \rangle,$$

for specific $M, g, h$ in the atlas.

Then implies

$$\langle g, Mh \rangle^2 \geq \langle g, Mg \rangle \langle h, Mh \rangle \quad \text{(hyperbolic ineq.)}$$

then implies

$$a_k^2 \geq a_{k+1} a_{k-1} \quad \text{(log-concave ineq.)}$$
The strategy

**Input:** Acyclic digraph $\mathcal{A}$, where each vertex $v$ is associated with
- $r \times r$ nonnegative symmetric matrix $M$,
- A nonnegative $r$-vector $h$.

**Goal:** Show every $M$ has hyperbolic inequality.

**Method:** Verify three conditions:
- Irreducibility condition;
- Inheritance condition;
- Subdivergence condition.
Irreducibility condition

- Matrix \( M \) associated to \( v \) is irreducible when restricted to its support;
- Vector \( h \) is associated to \( v \) is a positive vector.

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Inheritance condition

The $i$-th edge $e = (v, v_i)$ of $v$ is associated with linear map $T_i : \mathbb{R}^r \to \mathbb{R}^r$ such that, for every $x \in \mathbb{R}^r$,

$$i\text{-th coordinate of } Mx = \langle T_i x, M_i T_i h \rangle,$$

where $M$ and $h$ are associated to $v$, while $M_i$ is associated to $v_i$. 

![Diagram of inheritance condition](image)
Subdivergence condition

For every \( x \in \mathbb{R}^r \),

\[
\sum_{i=1}^{r} h_i \langle T_i x, M_i T_i x \rangle \geq \langle x, Mx \rangle,
\]

where \( h_i = i \)-th coordinate of \( h \).

**Note:** Often hardest condition to check, usually done through injective arguments.

**Note:** Equality occurs for matroids.
Bottom-to-top principle for inequalities

**Proposition**
Assume irreducibility, inheritance, subdivergence. If $M_1, \ldots, M_r$ has hyperbolic inequality property, then so does $M$.

Bottom-to-top principle reduces **Goal** to checking hyperbolic inequality only for **sink vertices**, which are usually **easy** to check.
Bottom-to-top principle
Bottom-to-top principle
Bottom-to-top principle
Bottom-to-top principle
How about equalities?
The strategy

**Input:**
- An acyclic digraph $\mathcal{A} := (\mathcal{V}, \mathcal{E})$ satisfying previous conditions;
- Vectors $g, h \in \mathbb{R}_{\geq 0}$;

**Goal:** Show "every" $M$ has hyperbolic equality,

$$\langle g, Mh \rangle^2 = \langle g, Mg \rangle \langle h, Mh \rangle.$$
Top-to-bottom principle for equalities

Proposition

Assume regularity condition. If $M$ has hyperbolic equality property, then so do $M_1, \ldots, M_r$.

Top-to-bottom principle expands hyperbolic equality to sink vertices, which usually gives combinatorial characterizations.
Top-to-bottom principle
Top-to-bottom principle
Top-to-bottom principle
Top-to-bottom principle
Conclusion

Problem: Log-concave inequalities and equalities.
Strategy:

- Build a combinatorial atlas;
- Verify the required conditions;
- Use hyperbolic inequality property to derive log-concave inequalities;
- Use hyperbolic equality to derive log-concave equalities.
THANK YOU!

Preprint to appear soon in your nearest arXiv server.

Webpage: http://math.ucla.edu/∼sweehong/

Email: sweehong@math.ucla.edu