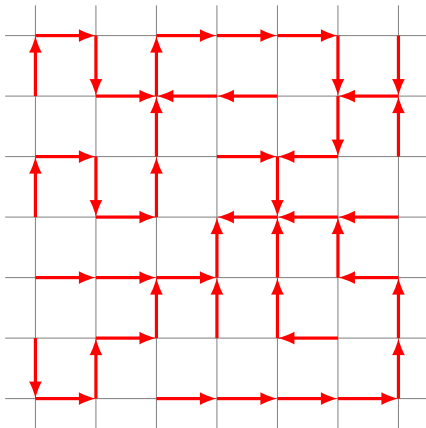


In between random walk and rotor walk

Swee Hong Chan

UCLA

Joint with Lila Greco, Lionel Levine, Peter Li







Random
walk

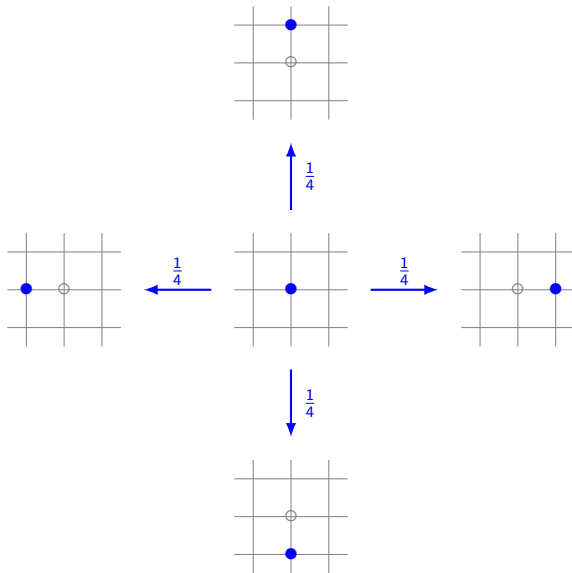


Rotor
walk

Simple random walk on \mathbb{Z}^2



Simple random walk on \mathbb{Z}^2



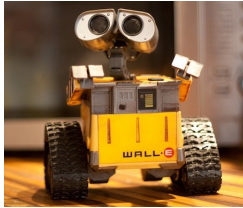
Simple random walk on \mathbb{Z}^2



- Visits every site infinitely often? **Yes!**
- Scaling limit? **The standard 2-D Brownian motion:**

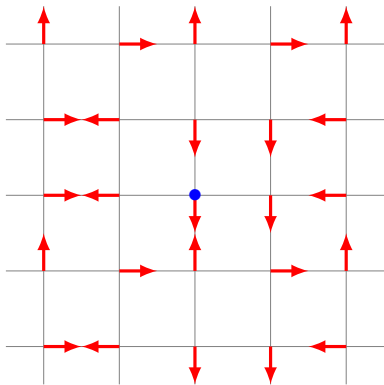
$$\underbrace{\left(\frac{1}{\sqrt{n}} X_{[nt]}\right)_{t \geq 0}}_{\text{location of the walker at time } [nt]} \xrightarrow{n \rightarrow \infty} \underbrace{\frac{1}{\sqrt{2}} (B_1(t), B_2(t))_{t \geq 0}}_{\text{independent standard Brownian motions}}.$$

Rotor walk on \mathbb{Z}^2



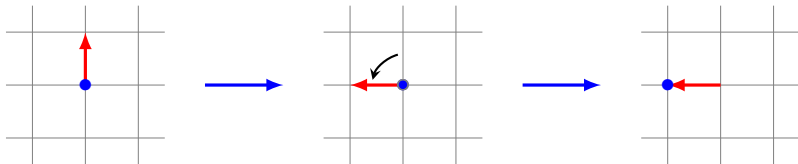
Rotor walk on \mathbb{Z}^2

Put a **signpost** at each site.



Rotor walk on \mathbb{Z}^2

Turn the signpost 90° counterclockwise, then follow the signpost.

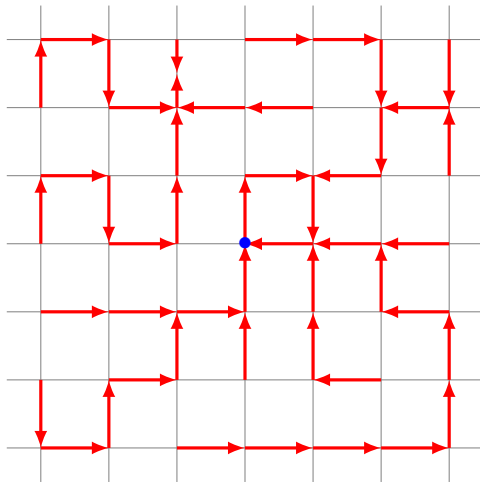


The signpost says:

“This is the way you went the last time you were here”,
(assuming you ever were!)

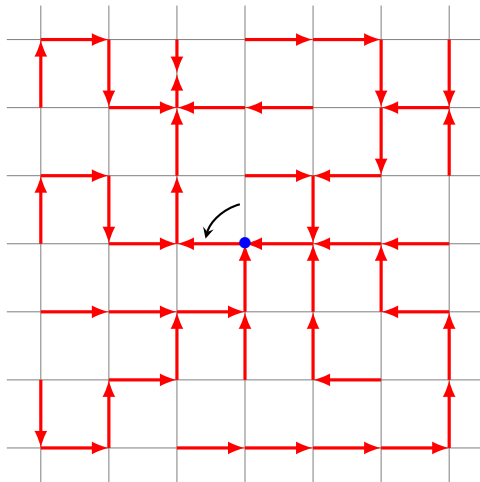
Rotor walk on \mathbb{Z}^2

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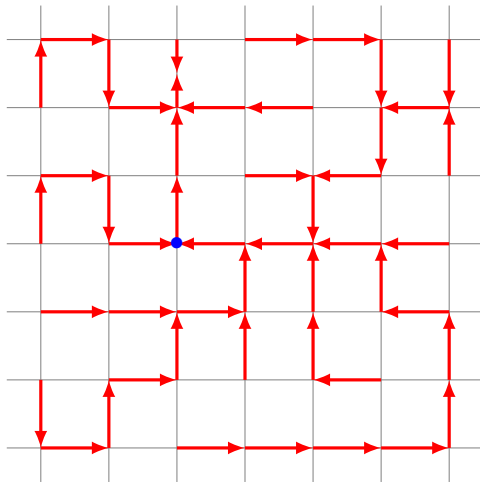
Rotor walk on \mathbb{Z}^2

Turn the signpost 90° counterclockwise, then follow the signpost.



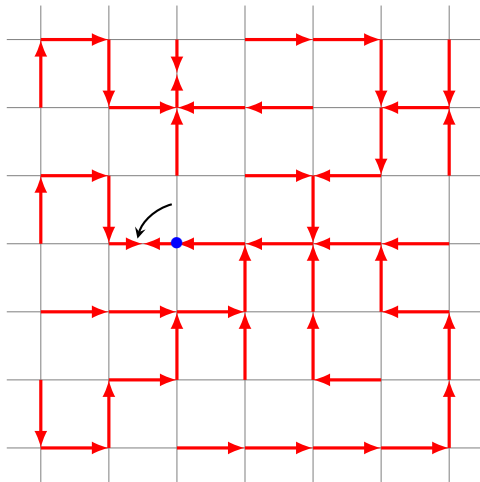
Rotor walk on \mathbb{Z}^2

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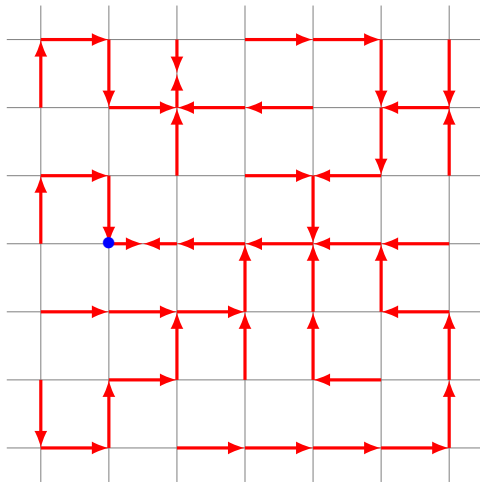
Rotor walk on \mathbb{Z}^2

Turn the signpost 90° counterclockwise, then follow the signpost.



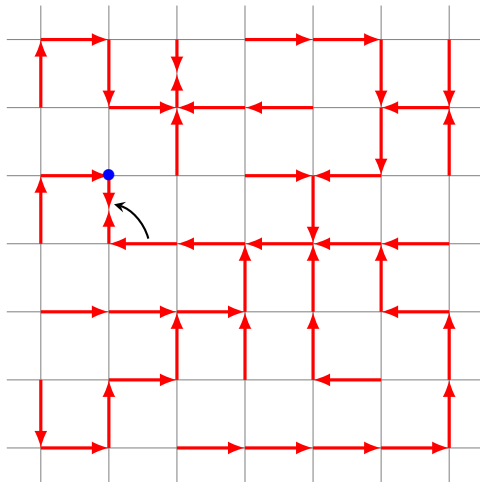
Rotor walk on \mathbb{Z}^2

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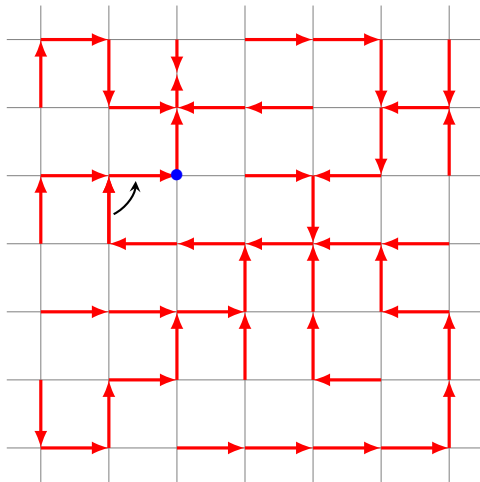
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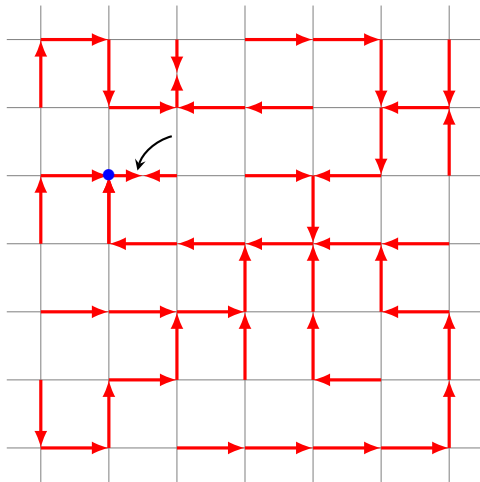
Rotor walk on \mathbb{Z}^2

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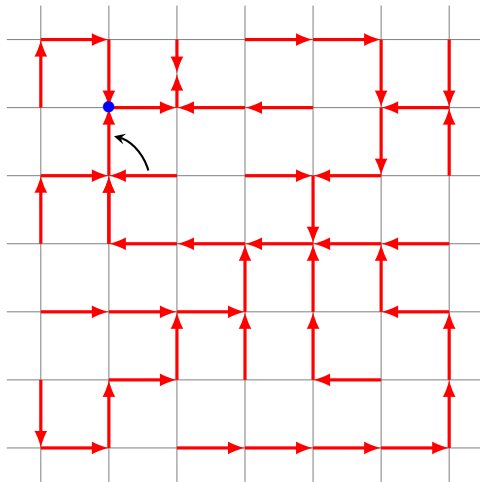
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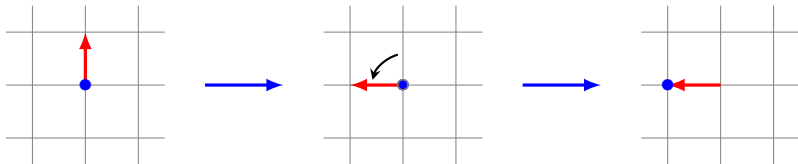
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Rotor walk on \mathbb{Z}^2

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The signpost says:

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(assuming you ever were!)

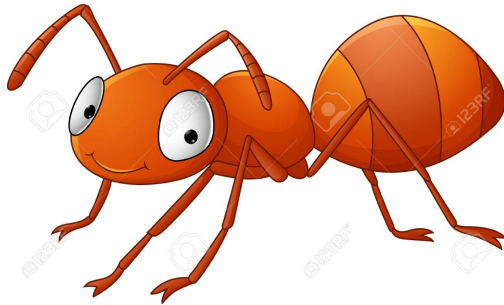
Why rotor walk?

Randomness can be (was) expensive to simulate!



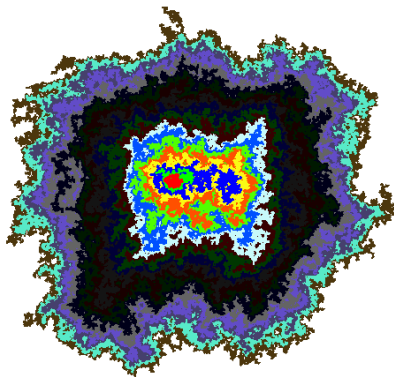
Why rotor walk?

As a model for ants' foraging strategy.



Why rotor walk?

As a model of self-organized criticality for statistical mechanics.



Visited sites after 80 returns to the origin (by Laura Florescu).

Conjectures for rotor walk on \mathbb{Z}^2



For initial signposts i.i.d. uniform among the four directions,

- (PDDK '96) Visits every site infinitely often?
- (PDDK '96) $\#\{X_1, \dots, X_n\}$ is $\asymp n^{2/3}$?
(compare with $n/\log n$ for the simple random walk.)
- (Kapri-Dhar '09) The asymptotic shape of $\{X_1, \dots, X_n\}$ is a disc?

More randomness please!

Well
studied



Many open
problems



Random

Deterministic

More randomness please!

Well
studied



Let's study
this!!!



Many open
problems

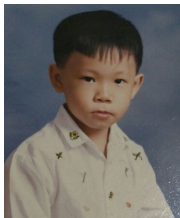


Random

Something
in between

Deterministic

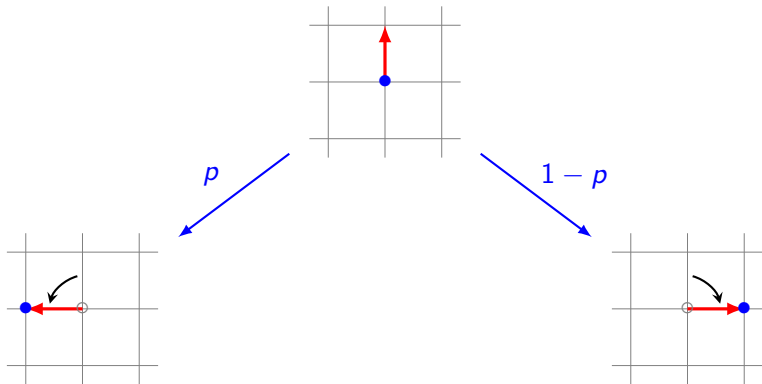
p -rotor walk on \mathbb{Z}^2



p -rotor walk on \mathbb{Z}^2

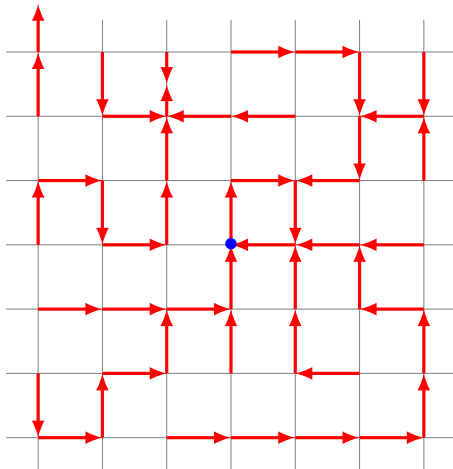
With probability p , turn the signpost 90° counter-clockwise.

With probability $1 - p$, turn the signpost 90° clockwise.



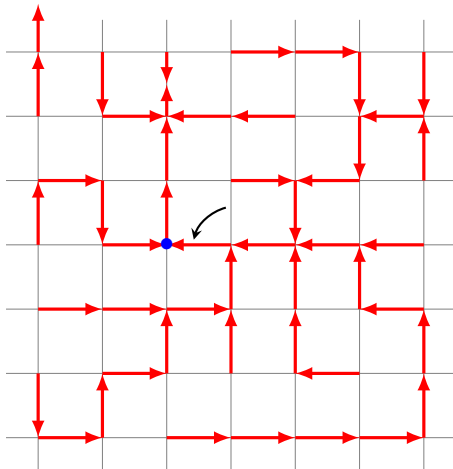
p -rotor walk on \mathbb{Z}^2

Follow rotor walk rule with prob. p , do the opposite with prob. $1 - p$



p -rotor walk on \mathbb{Z}^2

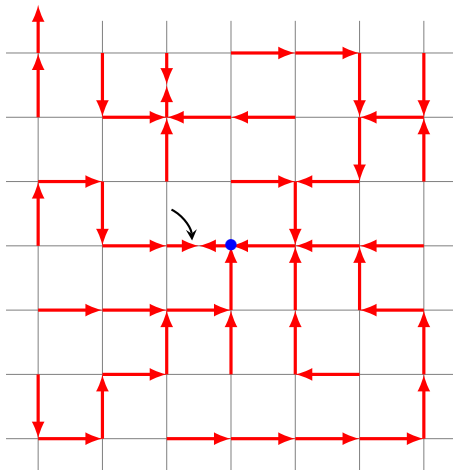
Follow rotor walk rule with prob. p , do the opposite with prob. $1 - p$



Follow the rule.

p -rotor walk on \mathbb{Z}^2

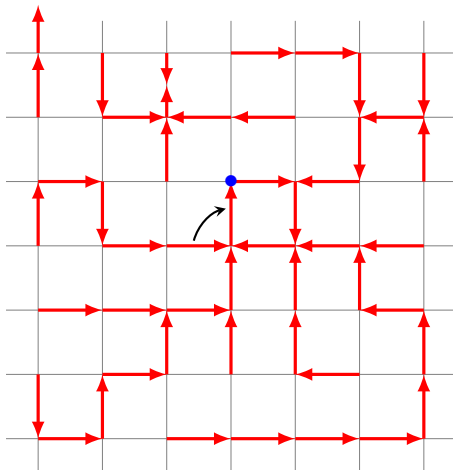
Follow rotor walk rule with prob. p , do the opposite with prob. $1 - p$



Do the opposite.

p -rotor walk on \mathbb{Z}^2

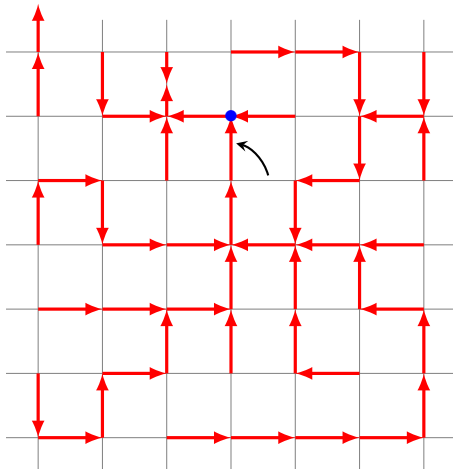
Follow rotor walk rule with prob. p , do the opposite with prob. $1 - p$



Do the opposite again.

p -rotor walk on \mathbb{Z}^2

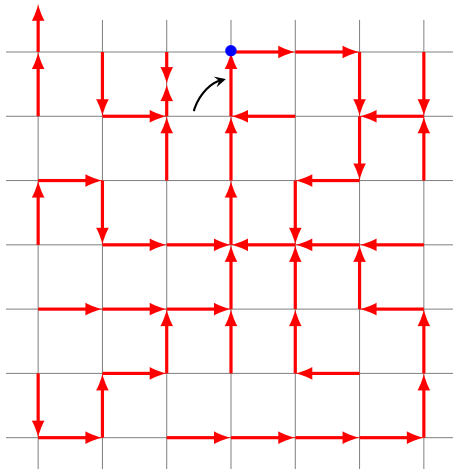
Follow rotor walk rule with prob. p , do the opposite with prob. $1 - p$



Follow the rule.

p -rotor walk on \mathbb{Z}^2

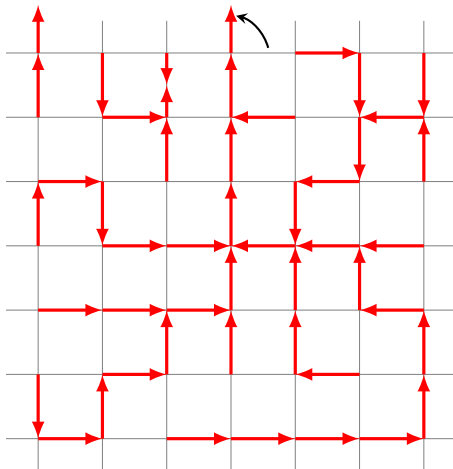
Follow rotor walk rule with prob. p , do the opposite with prob. $1 - p$



Do the opposite.

p -rotor walk on \mathbb{Z}^2

Follow rotor walk rule with prob. p , do the opposite with prob. $1 - p$

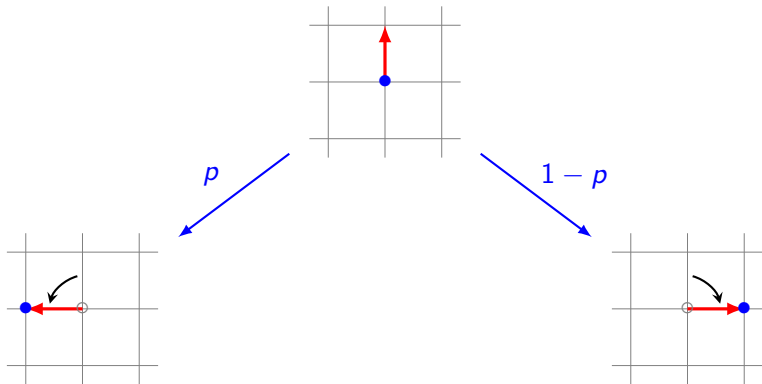


Ops...

p -rotor walk on \mathbb{Z}^2

With probability p , turn the signpost 90° counter-clockwise.

With probability $1 - p$, turn the signpost 90° clockwise.



Recover the rotor walk if $p = 1$.

Recurrence result for p-rotor walk

Recurrence for p -rotor walk on \mathbb{Z}^2

Theorem (C., 20+)

Let $p = \frac{1}{2}$ and let the *i.i.d uniform among four directions* be the initial signpost configuration. Then the p -rotor walk visits every vertex infinitely often almost surely.

Proof of recurrence for the simple random walk

Consider the following **martingale**:

$$M(t) := \underbrace{a(X(t))}_{\text{potential kernel}} - \underbrace{N(t)}_{\substack{\# \text{ of times} \\ \text{leaving } o}}.$$

Use the **optional stopping theorem**:

$$0 = \mathbb{E}[M(\underbrace{\tau(r)}_{\substack{\text{hitting time} \\ \text{of } \partial B_r \cup \{o\}}})] \approx \frac{2}{\pi} \ln r (1 - \underbrace{p_{\text{ret}}(r)}_{\substack{\text{prob. of return} \\ \text{before hitting } \partial B_r}}) - 1.$$

Proof of recurrence for the simple random walk (ctd.)

We rewrite the equation to

$$\underbrace{p_{\text{ret}}(r)}_{\substack{\text{prob. of return} \\ \text{before hitting } \partial B_r}} \approx 1 - \frac{\pi}{2 \ln r},$$

and we then conclude that

$$\underbrace{p_{\text{rec}}}_{\substack{\text{recurrence} \\ \text{probability}}} = 1 - \lim_{r \rightarrow \infty} \frac{\pi}{2 \ln r} = 1.$$

Proof of recurrence for p -rotor walk

Consider the following martingale:

$$M(t) := a(X(t)) - N(t) + \underbrace{\sum_{x \in \{X_0, \dots, X_t\}} w(x; \rho_t)}_{\text{compensator}}.$$

By the same argument as before,

$$\underbrace{p_{\text{rec}}}_{\text{recurrence probability}} = 1 - \lim_{r \rightarrow \infty} \frac{\pi}{2 \ln r} \left(\sum_{|x| \leq r} \mathbb{E}[w(x; \rho_{\tau(r)})] \right).$$

Proof of recurrence for p -rotor walk (ctd.)

We can estimate the terms in the compensator **locally** by

$$|\mathbb{E}[\mathbf{w}(x; \rho_{\tau(r)})]| \lesssim \left(1 - \frac{1}{2^{70}}\right) \frac{2}{\pi |x|^2}.$$

Plugging this estimate into previous equation,

$$p_{\text{rec}} \geq 1 - \lim_{r \rightarrow \infty} \frac{\pi}{2 \ln r} \left(\sum_{|x| \leq r} \left(1 - \frac{1}{2^{70}}\right) \frac{2}{\pi |x|^2} \right) = \frac{1}{2^{70}} > 0.$$

By **Kolmogorov zero-one law**, the recurrence probability is 1.

So we have proved ...

Theorem (C., 20+)

*Let $p = \frac{1}{2}$ and let the *i.i.d uniform among four directions* be the initial signpost configuration. Then the p -rotor walk visits every vertex infinitely often almost surely.*

The paper will be up in arXiv soon.

Send an email to sweehong@math.ucla.edu for early access.

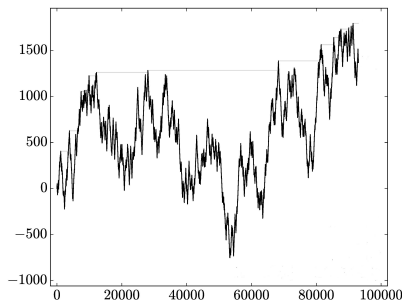


Scaling limit result for p-rotor walk

Scaling limit for p -rotor walk on \mathbb{Z}

(Huss, Levine, Sava-Huss 18) The scaling limit for p -rotor walk on \mathbb{Z} is a **perturbed Brownian motion** $(Y(t))_{t \geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\text{standard Brownian motion}} + \underbrace{a \sup_{0 \leq s \leq t} Y(s)}_{\text{perturbation at maximum}} + \underbrace{b \inf_{0 \leq s \leq t} Y(s)}_{\text{perturbation at minimum}}, \quad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

Scaling limit for p -rotor walk on \mathbb{Z}^2

Question: Is the scaling limit for p -rotor walk on \mathbb{Z}^2 a “2-D perturbed Brownian motion”?

Problem: How to define “2-D perturbed Brownian motion”?

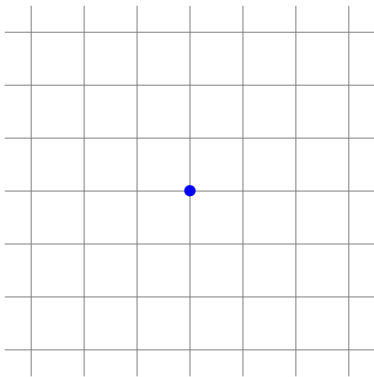
Scaling limit for p -rotor walk on \mathbb{Z}^2

Question: Is the scaling limit for p -rotor walk on \mathbb{Z}^2 a “2-D perturbed Brownian motion”?

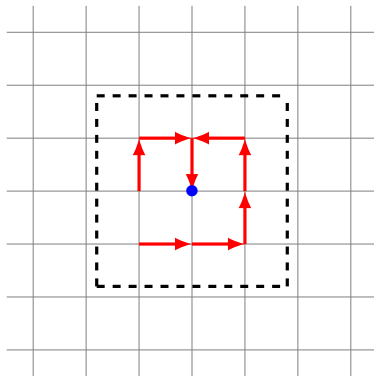
Problem: How to define “2-D perturbed Brownian motion”?

Conjecture: The scaling limit for p -rotor walk on \mathbb{Z}^2 when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

Uniform spanning forest plus one edge (USF^+)

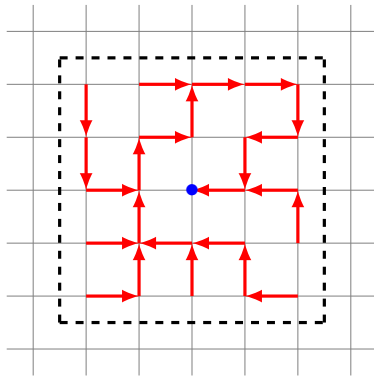


Uniform spanning forest plus one edge (USF^+)



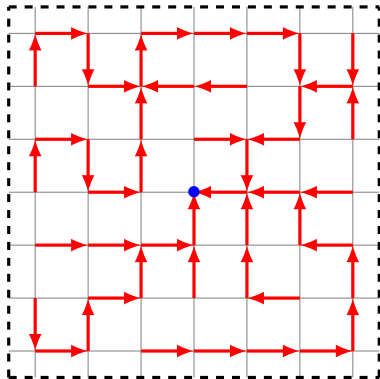
Pick a **spanning tree** of the black box directed to the origin (uniformly at random).

Uniform spanning forest plus one edge (USF^+)



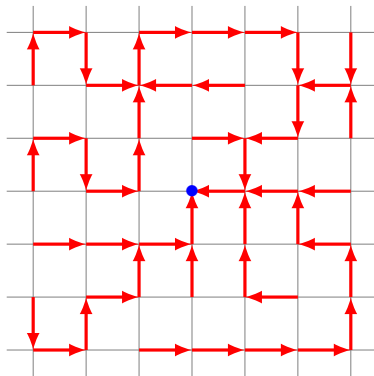
Take the limit as the black box grows until it covers \mathbb{Z}^2 .

Uniform spanning forest plus one edge (USF^+)



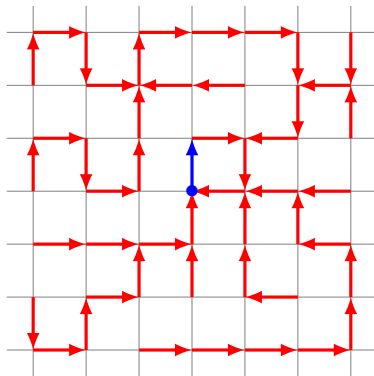
Take the limit as the black box grows until it covers \mathbb{Z}^2 .

Uniform spanning forest plus one edge (USF^+)



Take the limit as the black box grows until it covers \mathbb{Z}^2 .

Uniform spanning forest plus one edge (USF^+)



Add a **signpost** from the origin, uniform among the four directions.

Scaling limit for p -rotor walk on \mathbb{Z}^2

Theorem (C., Greco, Levine, Li '19+)

Let $p = \frac{1}{2}$ and let the *uniform spanning forest plus one edge* be the initial signpost configuration. Then, with probability 1, the p -rotor walk on \mathbb{Z}^2 scales to the standard 2-D Brownian motion:

$$\underbrace{\frac{1}{\sqrt{n}}(X_{[nt]})_{t \geq 0}}_{\text{location of the walker at time } [nt]} \xrightarrow{n \rightarrow \infty} \underbrace{\frac{1}{\sqrt{2}}(B_1(t), B_2(t))_{t \geq 0}}_{\text{independent Brownian motions}}.$$

Disclaimer: Proof in the paper was for *h-v walks*, not p -rotor walks.

Sketch of the scaling limit proof

Scaling limit

Martingale CLT

**Encounters vertical sign-
posts half the time a.s.**

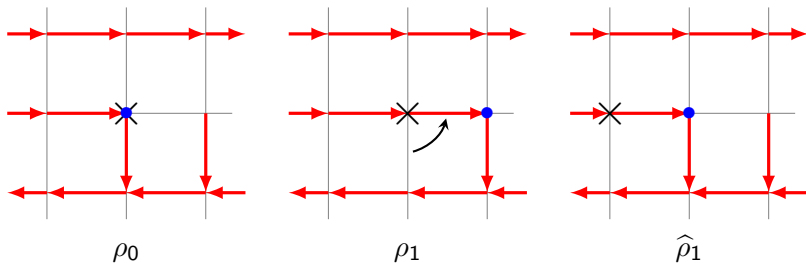
Ergodic theorem

**Stationarity and ergodicity
of USF^+ from walker's POV**

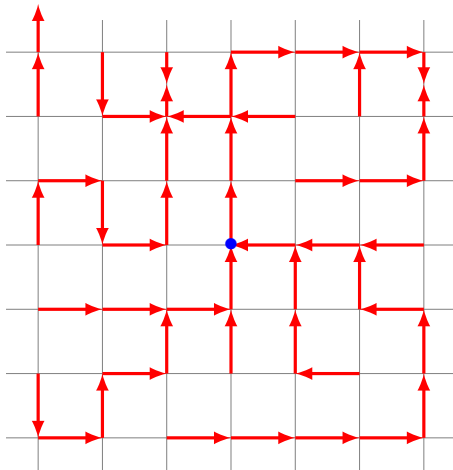
Stationarity from the walker's POV

A signpost configuration $(\rho_0(x))_{x \in \mathbb{Z}^2}$ is **stationary in time from the walker's point of view** if

$$\underbrace{(\hat{\rho}_1(x))_{x \in \mathbb{Z}^2}}_{\text{signpost conf. at time 1 from walker's POV}} := (\rho_1(x - X_1))_{x \in \mathbb{Z}^2} \stackrel{d}{=} \underbrace{(\rho_0(x))_{x \in \mathbb{Z}^2}}_{\text{signpost conf. at time 0}}$$

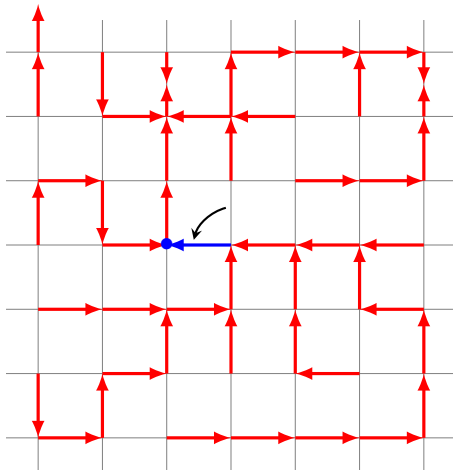


Why is USF^+ stationary?



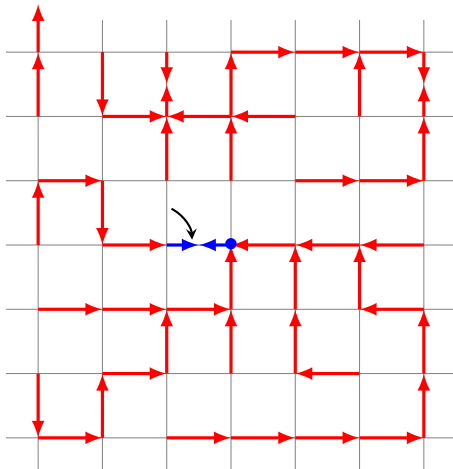
The signposts at previously visited sites form a **tree** oriented toward the walker.

Why is USF^+ stationary?



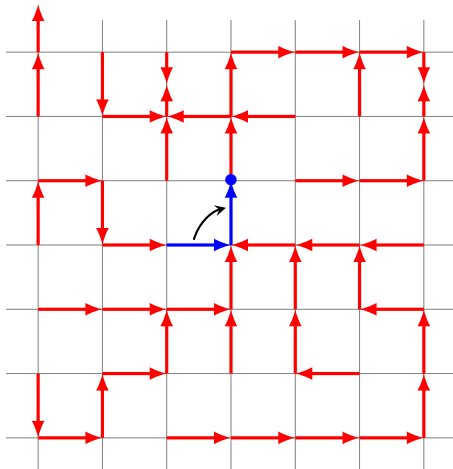
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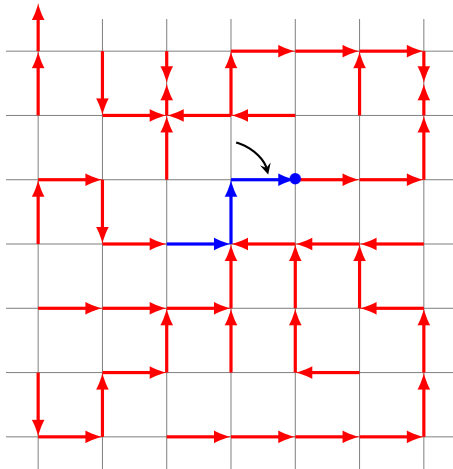
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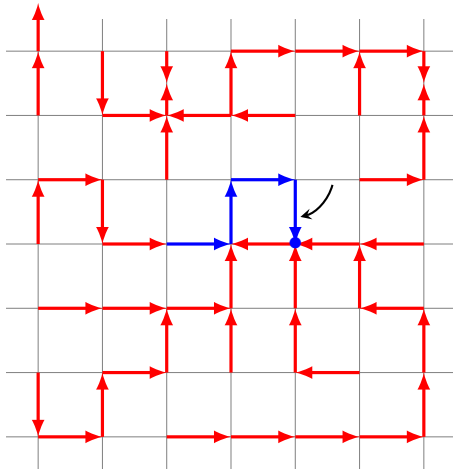
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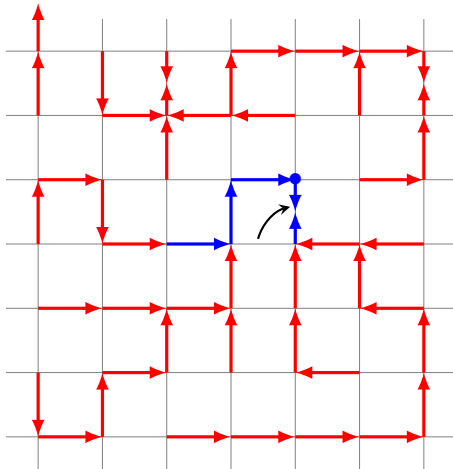
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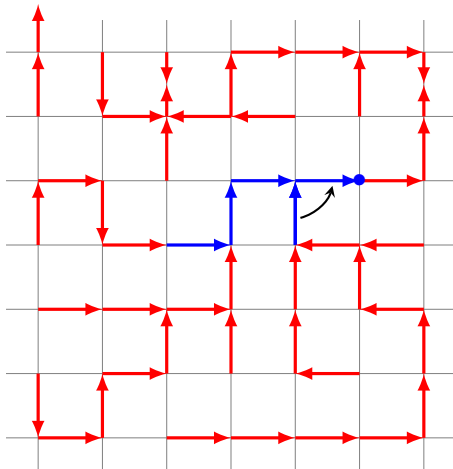
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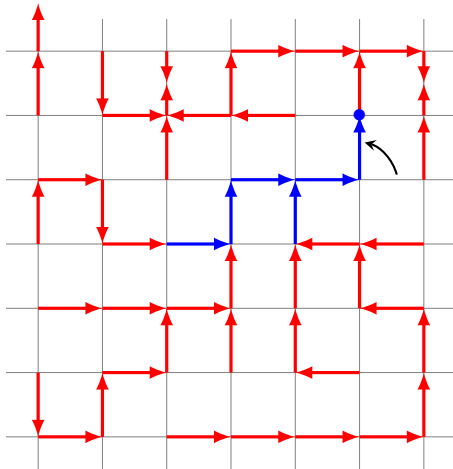
The signposts at previously visited sites form a **tree** oriented toward the walker.

Why is USF^+ stationary?



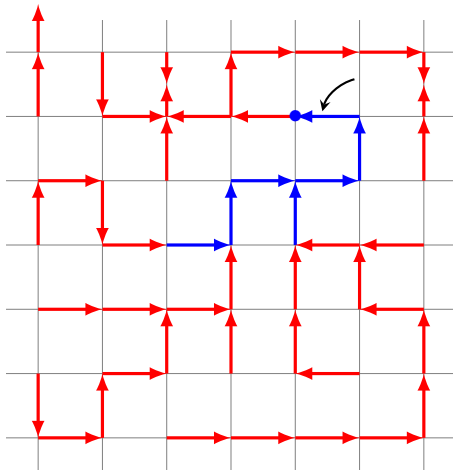
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Why is USF^+ stationary?



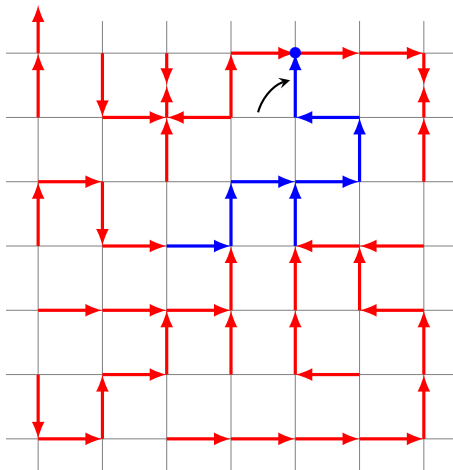
The signposts at previously visited sites form a **tree** oriented toward the walker.

Why is USF^+ stationary?



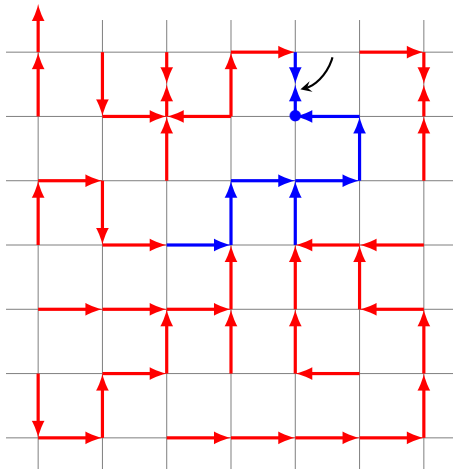
The signposts at previously visited sites form a **tree** oriented toward the walker.

Why is USF^+ stationary?



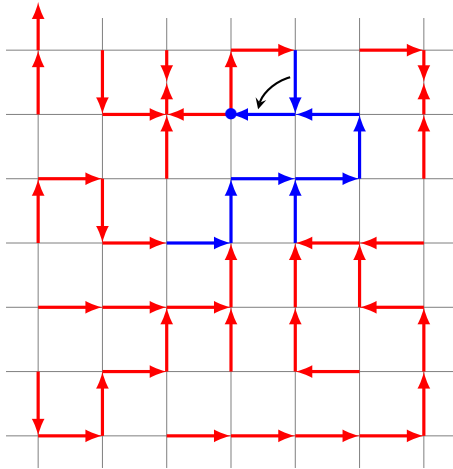
The signposts at previously visited sites form a **tree** oriented toward the walker.

Why is USF^+ stationary?



The signposts at previously visited sites form a **tree** oriented toward the walker.

Why is USF^+ stationary?



The signposts at previously visited sites form a **tree** oriented toward the walker.

So we have proved...

Theorem (C., Greco, Levine, Li '19+)

Let $p = \frac{1}{2}$ and let the *uniform spanning forest plus one edge* be the initial signpost configuration. Then, with probability 1, the p -rotor walk on \mathbb{Z}^2 scales to the standard 2-D Brownian motion:

$$\underbrace{\frac{1}{\sqrt{n}}(X_{[nt]})_{t \geq 0}}_{\text{location of the walker at time } [nt]} \xrightarrow{n \rightarrow \infty} \underbrace{\frac{1}{\sqrt{2}}(B_1(t), B_2(t))_{t \geq 0}}_{\text{independent Brownian motions}}.$$



Back to our motivation

Well
studied



Simple
random walk

Know a
little bit now



p -rotor
walk

Many open
problems



Rotor walk

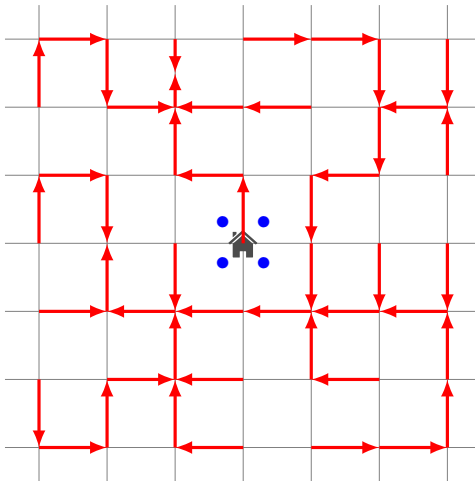
Let's apply what we have learnt to rotor walk.

Escape rate of rotor walk



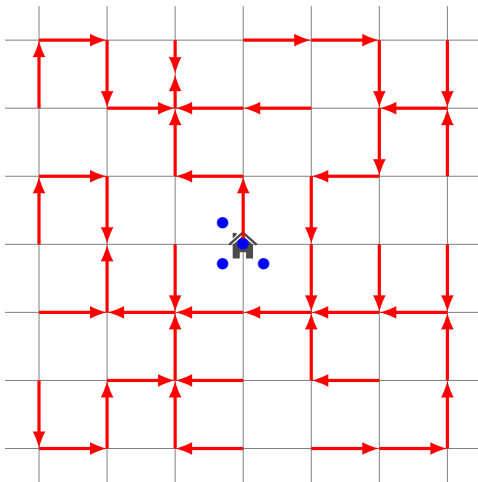
Prison break using rotor walk

Put n walkers at the origin (the prison).



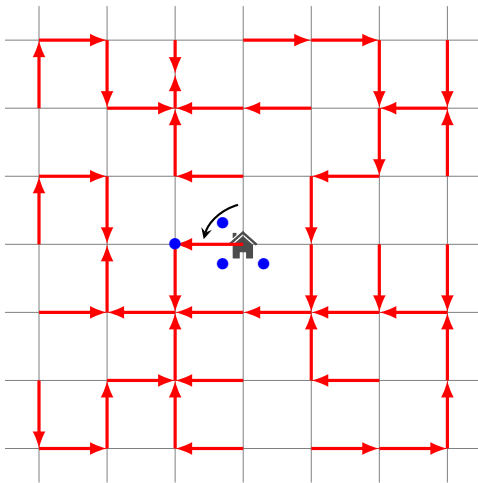
Prison break using rotor walk

First walker performs rotor walk, remove if returns to prison.



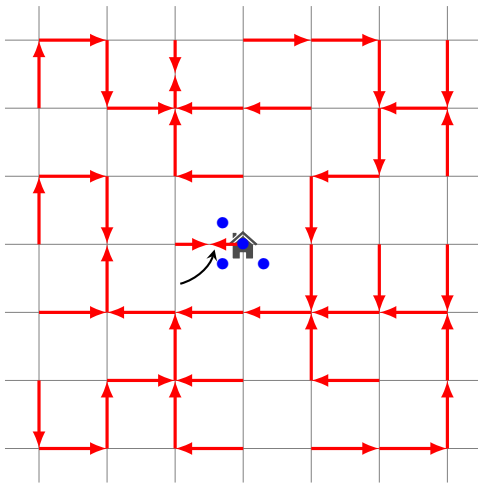
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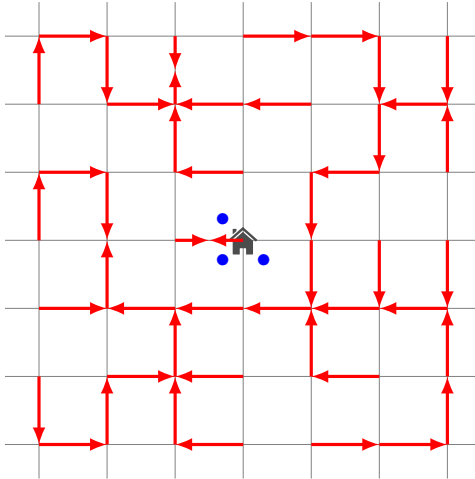
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First walker performs rotor walk, remove if returns to prison.



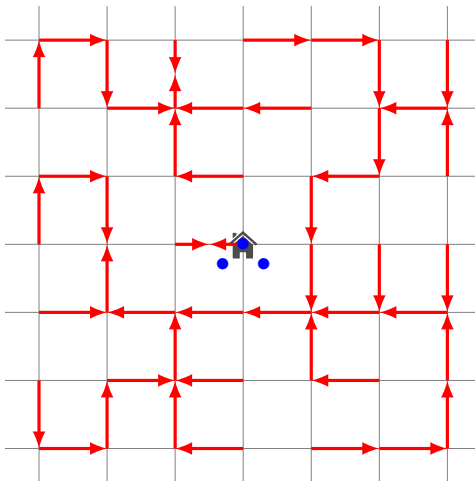
Prison break using rotor walk

First walker returns to prison, and is removed.



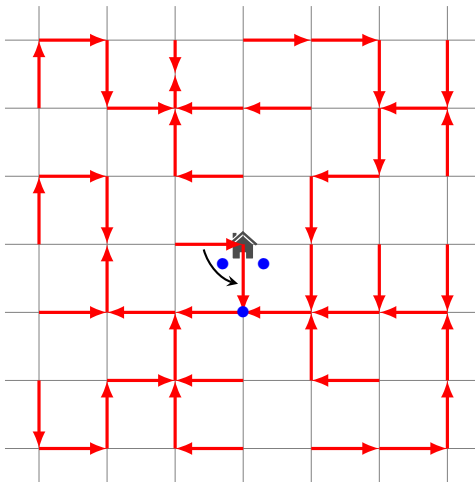
Prison break using rotor walk

Second walker performs rotor walk, remove if returns to prison.



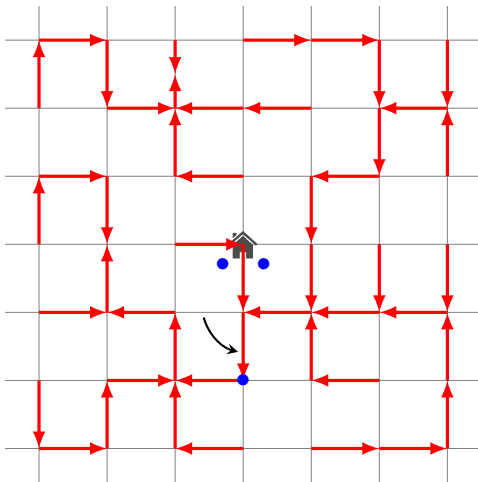
Prison break using rotor walk

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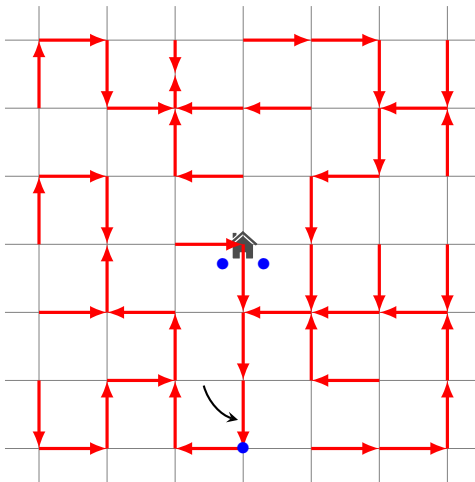
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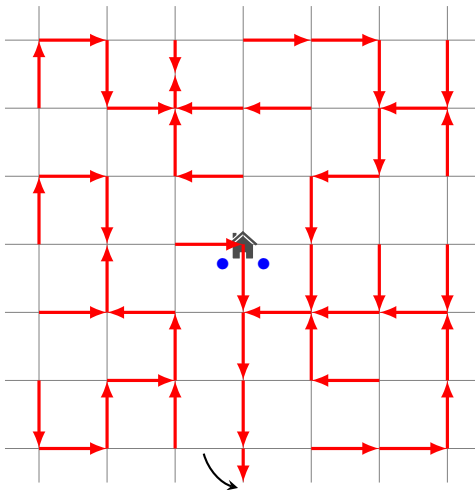
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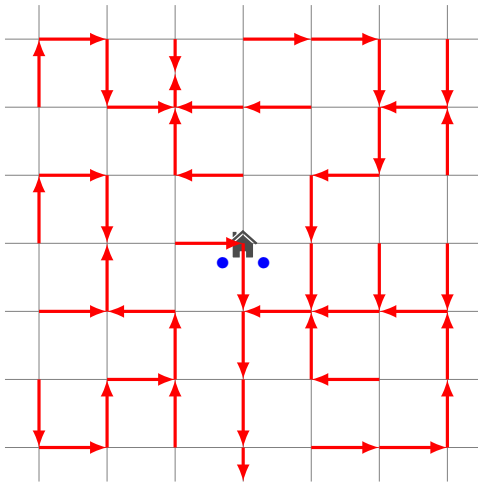
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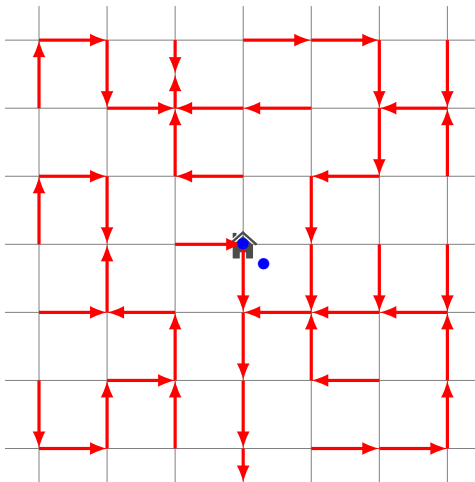
Prison break using rotor walk

Second walker never returns to origin.



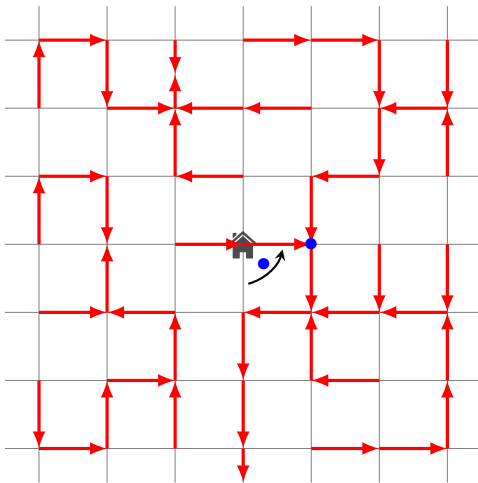
Prison break using rotor walk

Third walker performs rotor walk, remove if returns to prison.



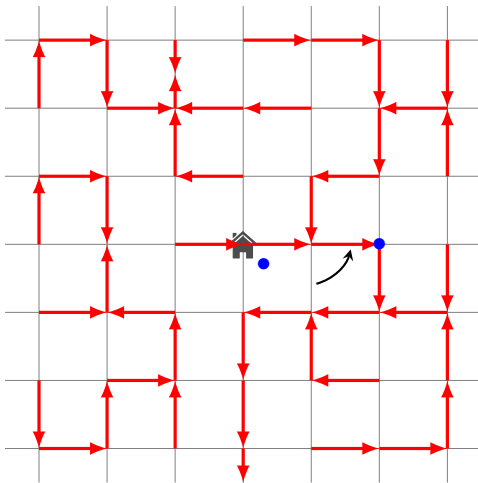
Prison break using rotor walk

Third walker performs rotor walk, remove if returns to prison.



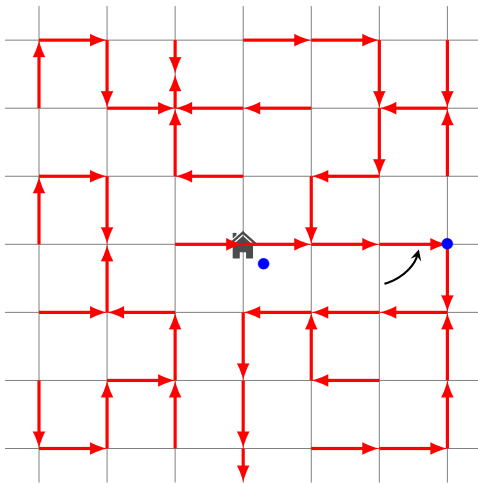
Prison break using rotor walk

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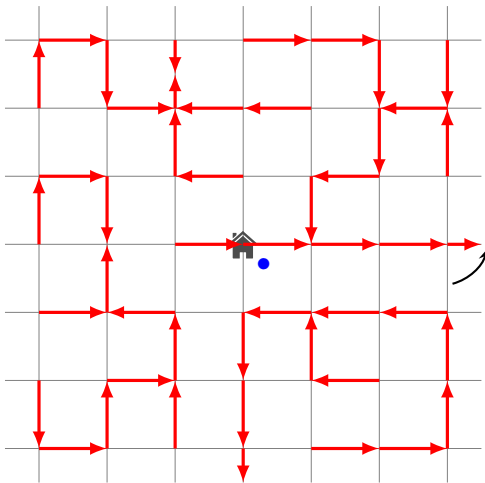
Prison break using rotor walk

Third walker performs rotor walk, remove if returns to prison.



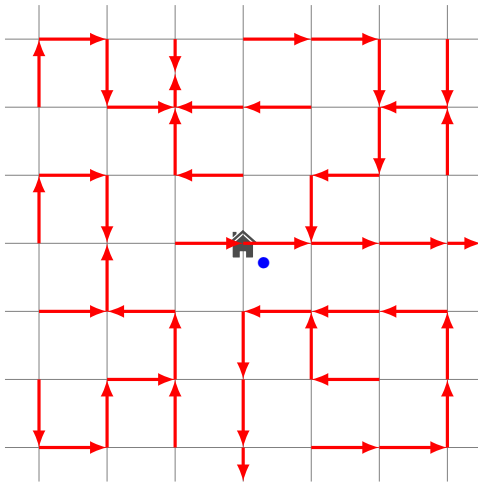
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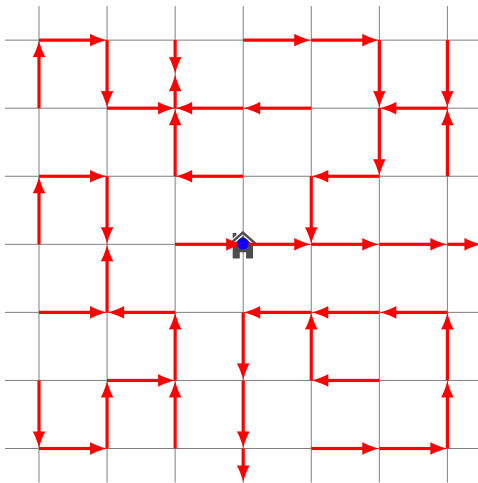
Prison break using rotor walk

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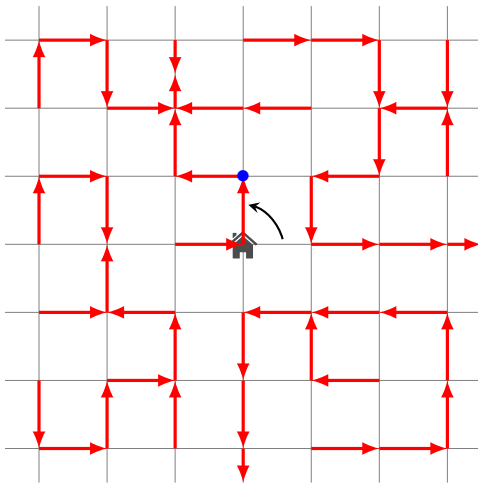
Prison break using rotor walk

Fourth walker performs rotor walk, remove if returns to prison.



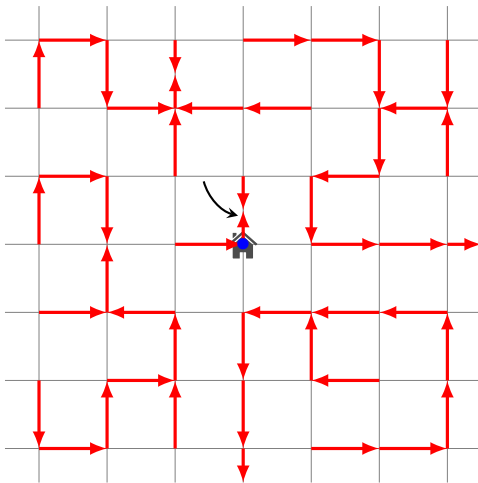
Prison break using rotor walk

Fourth walker performs rotor walk, remove if returns to prison.



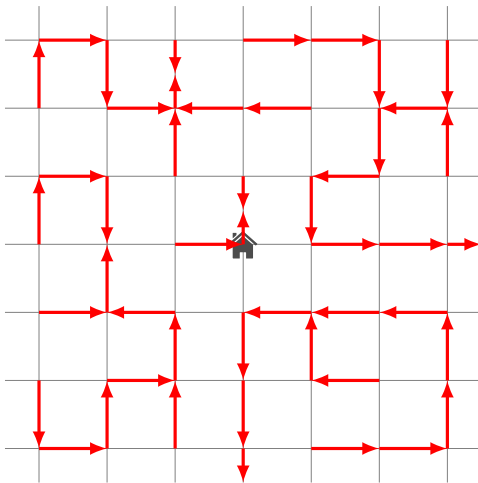
Prison break using rotor walk

Fourth walker performs rotor walk, remove if returns to prison.



Prison break using rotor walk

Fourth walker returns to prison, and is removed.



Escape rate of rotor walk



The **escape rate** of n rotor walkers with initial signpost ρ is

$$r_{\text{esc}}(\rho, n) := \frac{\text{number of escaped walkers}}{n}.$$

The **escape rate of rotor walk** is a deterministic counterpart of the escape probability of simple random walk.

What was known about escape rate

Theorem (Schramm '10 (posthumous))

For *any* initial signpost ρ ,

$$\limsup_{n \rightarrow \infty} \underbrace{r_{\text{esc}}(\rho, n)}_{\substack{\text{escape rate} \\ \text{of rotor walk}}} \leq \underbrace{p_{\text{esc}}(\text{SRW})}_{\substack{\text{escape prob.} \\ \text{of SRW}}}.$$

Corollary

On \mathbb{Z}^2 , for *any* initial signpost ρ ,

$$\lim_{n \rightarrow \infty} r_{\text{esc}}(\rho, n) = p_{\text{esc}}(\text{SRW}) = 0.$$

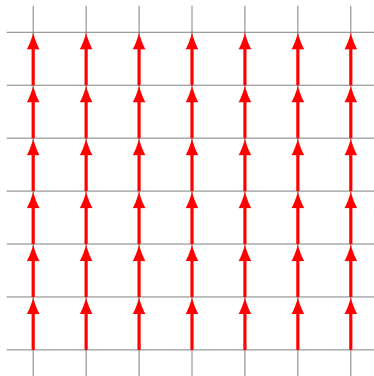
In fact, this is true for all *recurrent* graphs.

What was known about escape rate

Theorem (Florescu Ganguly Levine Peres '13)

On \mathbb{Z}^d with $d \geq 3$, for the *one-directional* initial signpost ρ ,

$$\liminf_{n \rightarrow \infty} r_{\text{esc}}(\rho, n) > 0.$$



Escape rate conjecture

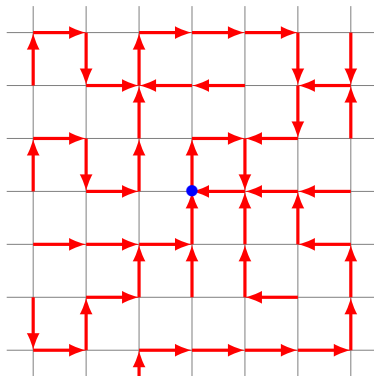
Conjecture (FGLP '13)

For *any transient* graph, there *exists* an initial signpost ρ for which

$$\lim_{n \rightarrow \infty} r_{\text{esc}}(\rho, n) = p_{\text{esc}}(\text{SRW}).$$

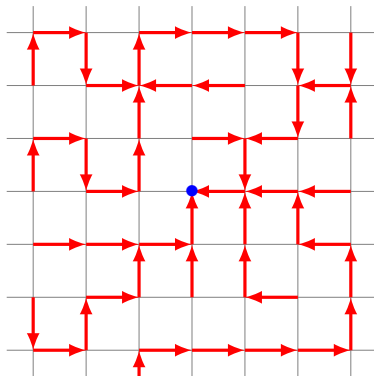


Uniform spanning forest oriented to infinity (USF^∞)



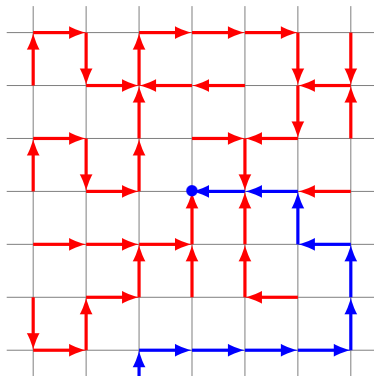
Start with uniform spanning forest plus one edge from before.

Uniform spanning forest oriented to infinity (USF^∞)



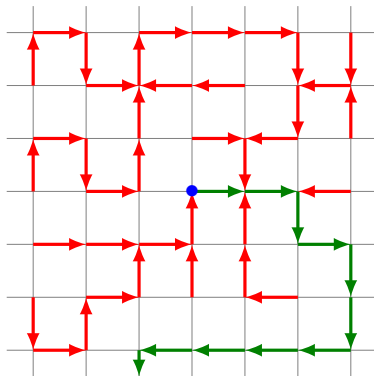
Remove the signpost at the origin.

Uniform spanning forest oriented to infinity (USF^∞)



Find the unique **infinite path** oriented to origin.

Uniform spanning forest oriented to infinity (USF^∞)



Reverse the orientation of this infinite path.

Answering the escape rate conjecture

Theorem (C. '19)

On \mathbb{Z}^d , almost every ρ sampled from USF^∞ satisfies

$$\lim_{n \rightarrow \infty} r_{\text{esc}}(\rho, n) = p_{\text{esc}}(\text{SRW}).$$

This is proved using **infinite-step stationarity** for rotor walk on \mathbb{Z}^d .

Except that ...

- The conjecture of FGLP '13 is for **all transient graphs**;
- There are already other constructions for the **special case** of \mathbb{Z}^d (He '14) and trees (Angel Holroyd '11);
- Our construction of the initial signpost ρ is **not deterministic**.

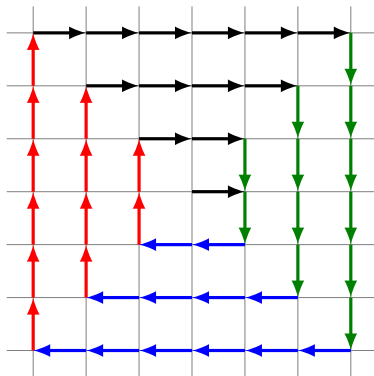


Complete answer to the escape rate conjecture

Theorem (C., '20)

For *any transient graph*, the initial signpost ρ_{\max} satisfies

$$\lim_{n \rightarrow \infty} r_{\text{esc}}(\rho_{\max}, n) = p_{\text{esc}}(\text{SRW}).$$



Escape rate formula

Lemma

For any initial signpost ρ and number of walkers n ,

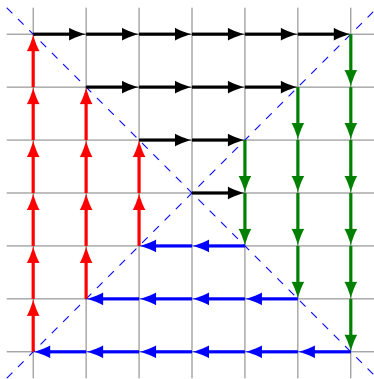
$$r_{\text{esc}}(\rho, n) = p_{\text{esc}}(\text{SRW}) - \sum_{x \in \mathbb{Z}^d} \left(\underbrace{w_x[\rho_n(x)]}_{\text{signpost at } x \text{ after } n\text{-th walk}} - \underbrace{w_x[\rho(x)]}_{\text{initial signpost at } x} \right),$$

where w_x is a local compensator term.

The formula is inspired by the [martingale](#) used in proving recurrence for p -rotor walk.

Our initial signpost configuration

The configuration ρ_{\max} is constructed by choosing, for each x , the direction $\rho_{\max}(x)$ that maximizes compensator w_x .



Proof of the escape rate conjecture

- By the [escape rate formula](#),

$$r_{\text{esc}}(\rho, n) = p_{\text{esc}}(SRW) - \sum_{x \in \mathbb{Z}^d} \left(\mathbf{w}_x[\rho_n(x)] - \mathbf{w}_x[\rho(x)] \right),$$

- By our choice of [\\(\rho_{\max}\\)](#),

$$r_{\text{esc}}(\rho_{\max}, n) \geq p_{\text{esc}}(SRW).$$

- On the other hand, [Schramm's inequality](#) gives us

$$\limsup_{n \rightarrow \infty} r_{\text{esc}}(\rho_{\max}, n) \leq p_{\text{esc}}(SRW).$$

- Hence,

$$\lim_{n \rightarrow \infty} r_{\text{esc}}(\rho_{\max}, n) = p_{\text{esc}}(SRW).$$

So we have proved...

Theorem (C., '20)

For *any transient graph*, the initial signpost ρ_{\max} satisfies

$$\lim_{n \rightarrow \infty} r_{\text{esc}}(\rho_{\max}, n) = p_{\text{esc}}(\text{SRW}).$$



What is next?

Conjecture

Let $p \neq \frac{1}{2}$. Prove that p -rotor walk with i.i.d. uniform signpost configuration is *recurrent*.

Obstacle: Need a *good estimate* for the compensator.

$$\underbrace{M(t)}_{\text{martingale}} := a(X(t)) - N(t) + \underbrace{\sum_{x \in \{X_0, \dots, X_t\}} w(x; \rho_t)}_{\text{compensator}}.$$

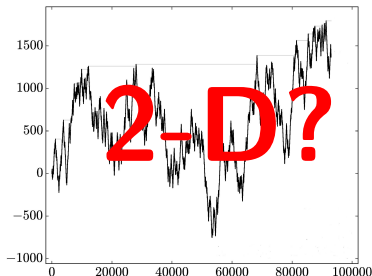


What is next?

Problem

Find the *scaling limit* for the p -rotor walk with i.i.d. uniform signpost configuration.

Obstacle: Need to define “2-D perturbed Brownian motion (?)”.



What is next?

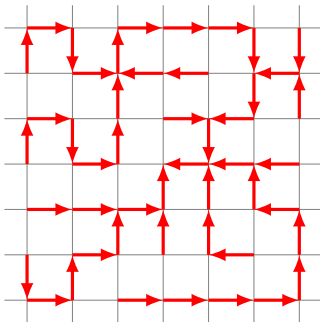
Conjecture

*For any graph, the i.i.d. uniform signpost configuration has rotor walk **escape rate** equal to the escape probability of the SRW, i.e.,*

$$\lim_{n \rightarrow \infty} r_{\text{esc}}(\rho, n) = p_{\text{esc}}(\text{SRW}).$$

So far has only been proved for regular trees (Angel Holroyd '11).



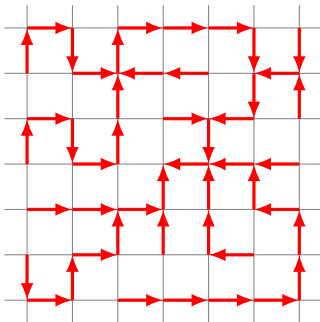


(Some of) the preprints can be found in my webpage:

<http://math.ucla.edu/~sweehong/>

My email: sweehong@math.ucla.edu

THANK YOU!



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