Math 170E

Lecture Notes Section 5.8 [*,†]

Chebyshev's inequality and Markov's inequality

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

[*]Version date: Friday 12th March, 2021, 01:30.

[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Probability inequalities: Example

Let X be the systolic blood pressure of the friendly instructor, which is a random variable with mean 120 and variance 16.

The safe range for his blood pressure is strictly between 108 and 132.

What is the probability that his blood pressure tomorrow is within the safe range?

The question is asking for the probability

$$P(108 < X < 132).$$

We **cannot** compute this probability since we do not know what the random variable X is.

Now, suppose that the doctor recommends no further need for medication if this probability is greater than 80%.

Without knowing the random variable, can the instructor conclude that he does not need further medication?

2 Markov inequality: Theorem

Theorem 1. *Let X be* ***any*** *nonnegative random variable. Then, for any real number $a > 0$,*

$$P(X \geq a) \quad \leq \quad \frac{E[X]}{a}.$$

Proof: We have

$$\begin{aligned} E[X] \quad &= \quad E[X, X < a] \;+\; E[X, X \geq a] \\ \geq \quad & E[0, X < a] \;+\; E[X, X \geq a] \quad \text{(because } X \text{ is nonnegative)} \\ \geq \quad & E[0, X < a] \;+\; E[a, X \geq a] \\ = \quad & 0 \;+\; a\,P(X \geq a), \end{aligned}$$

for which the theorem now follows.

3 Chebyshev's inequality

Theorem 2. *Let X be* ***any*** *random variable with mean μ and variance σ^2. Then, for any real $k \geq 0$,*

$$P\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq \frac{1}{k^2}.$$

Proof: Let Z be the random variable $Z = \left(\frac{X-\mu}{\sigma}\right)^2$.

Then Z is a nonnegative RV with mean $E[Z] = 1$.

We then have

$$P\left(\frac{|X-\mu|}{\sigma} \geq k\right) = P\left(\left(\frac{X-\mu}{\sigma}\right)^2 \geq k^2\right)$$

$$= P(Z \geq k^2) \leq \frac{1}{k^2} \quad \text{(Markov's inequality)}.$$

4 Probability inequalities: Answer

The question is asking for a lower bound for $P(108 < X < 132)$. Note that

$$\begin{aligned}
X = 108 \quad &\Rightarrow \quad \frac{X-\mu}{\sigma} = \frac{108-120}{\sqrt{16}} = -3; \\
X = 132 \quad &\Rightarrow \quad \frac{X-\mu}{\sigma} = \frac{132-120}{\sqrt{16}} = 3.
\end{aligned}$$

Hence we have

$$\begin{aligned}
P(108 < X < 132) &= P\left(\frac{|X-\mu|}{\sigma} < 3\right) \\
&= 1 - P\left(\frac{|X-\mu|}{\sigma} \geq 3\right) \geq 1 - \frac{1}{3^2} \quad \text{(Chebyshev)} \\
&= 0.889.
\end{aligned}$$

Thus the instructor does not need to take medication.

5 Probability inequalities: Application

Chebyshev's and Markov's inequalities are often used in the situation when **the random variable is unknown**.

However, if **the random variable is known**, it is usually better to compute the probability directly.

6 Probability inequality: Example 2

Let X be the **normal** random variable with mean 120 and variance 16.

Calculate the probability $P(108 < X < 132)$.

Answer: We have

$$
\begin{aligned}
P(108 < X < 132) &= P\left(-3 < \frac{X-\mu}{\sigma} < 3\right) \\
&= \Phi(3) - \Phi(-3) \\
&= 2\Phi(3) - 1 = 0.9974,
\end{aligned}
$$

which is much better than the lower bound from Chebyshev's inequality.

7 Weak law of large numbers

One application of Chebyshev's inequality is the following theorem.

Theorem 3. *Let $X_1, X_2, \ldots$ be independent random variables with mean μ and variance σ^2. Let*

$$\overline{\mathrm{X}} \quad = \quad \frac{X_1 + \ldots + X_n}{n}.$$

Then, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P\Big(|\,\overline{\mathrm{X}} - \mu| \geq \varepsilon\,\Big) \quad = \quad 0.$$

Intuitively, this means that, for sufficiently large n, the random variables $\overline{\mathrm{X}}$ loses **almost all** of its randomness and is **essentially equal to** μ.

8 Weak law of large numbers: Proof

Recall from extra weak law of large number that

$$E[\overline{\mathrm{X}}] \quad = \quad \mu; \qquad \mathrm{Var}[\overline{\mathrm{X}}] \quad = \quad \frac{\sigma^2}{n}.$$

By Chebyshev's inequality,

$$\begin{aligned} P\big(|\,\overline{\mathrm{X}} - \mu| \geq \varepsilon\,\big) \quad &= \quad P\left(\frac{|\,\overline{\mathrm{X}} - \mu|}{\sigma/\sqrt{n}} \geq \frac{\varepsilon}{\sigma/\sqrt{n}}\right) \\ &\leq \quad \frac{\sigma^2}{\varepsilon\, n^2}. \end{aligned}$$

Taking the limit of the equation above,

$$\lim_{n\to\infty} P\big(|\,\overline{\mathrm{X}} - \mu| \geq \varepsilon\,\big) \quad = \quad 0,$$

as desired.

9 Strong law of large numbers

There exists **strong law of large numbers**, which contains the same intuition, but with a stronger form of convergence (**almost sure convergence**).

The strong version is taught in graduate level probability course.