Math 170E

Lecture Notes Section 5.6 [*†]

Central limit theorem

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Super weak law of large numbers

Recall from Section 5.3 the following theorem.

Theorem 1. *Let $X_1, X_2, \ldots$ be independent random variables with mean μ and variance σ^2. Let $\overline{\mathrm{X}}$ be the sampled mean*

$$\overline{\mathrm{X}} := \frac{X_1 + \ldots + X_n}{n}.$$

Then,

$$E[\overline{\mathrm{X}}] = \mu; \qquad \mathrm{Var}[\overline{\mathrm{X}}] = \frac{\sigma^2}{n}.$$

So, as $n \to \infty$, the average $\overline{\mathrm{X}}$ gets closer and closer to the mean μ.

2 Law of large numbers: Intuition, stock

One can think of X_i as the profit of a stock investment at the i-th year.

When investing, we need to pay attention to both **profit** and **risk**.

Then $E[\overline{\mathrm{X}}] = \mu$ is the **expected profit** if you invest for n years on average.

Furthermore, $\mathrm{Var}[\overline{\mathrm{X}}]$ measures the **expected risk** of your investment for n years.

Typical question when investing:

"**How much money will I make** if I invest in n years?"

This question cannot be answered with certainty, since no one can predict the future. A better question is

"**What is the probability** that I will make x amount of money if I invest in n years?"

This question can be answered by the **central limit theorem**.

3 Central limit theorem: Example

Let $\overline{\mathrm{X}}$ be the average of the profit of your company for $n = 25$ years. Suppose that $X_1, \dots, X_n$ has mean 15 (millions USD) and variance 4 (millions USD).

Approximate the probability

$$P(14.4 < \overline{\mathrm{X}} < 15.6).$$

4 Central limit theorem

Theorem 2. *Let $X_1, X_2, \ldots$ be independent random variables with mean μ and variance σ^2. Let W be the random variable*

$$W = \frac{\overline{\mathrm{X}} - \mu}{\sigma/\sqrt{n}} = \frac{(X_1 + \ldots + X_n) - n\,\mu}{\sigma\sqrt{n}}.$$

Then the cdf of W converges to the cdf of standard normal random variable as $n \to \infty$.

That is to say, when n is large, we have for all real w,

$$P(W \leq w) \quad \approx \quad \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz \quad = \quad \Phi(w).$$

where $\Phi(\cdot)$ is the cdf of standard normal RV from Table Va. This is the reason why normal RVs and bell curves are so important.

5 Central limit theorem: Answer

Answer: Let $W = \frac{\overline{\mathrm{X}} - \mu}{\sigma/\sqrt{n}}$ be as in the central limit theorem. Then we have

$$\overline{\mathrm{X}} = 14.4 \qquad \Longrightarrow \qquad W = \frac{(14.4) - (15)}{(2)/\sqrt{(25)}} = \frac{-0.6}{0.4} = -1.5;$$

$$\overline{\mathrm{X}} = 15.6 \qquad \Longrightarrow \qquad W = \frac{(15.6) - (15)}{(2)/\sqrt{(25)}} = \frac{0.6}{0.4} = 1.5.$$

By the central limit theorem, we then have

$$\begin{aligned} P(14.4 < \overline{\mathrm{X}} < 15.6) &= P(-1.5 < W < 1.5) \\ \approx \quad \Phi(1.5) - \Phi(-1.5) &= 2\,\Phi(1.5) - 1 \\ = \quad 0.8664. \end{aligned}$$