Math 170E

Lecture Notes Section 5.4 [*,†]

Moment generating functions

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†] This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Moment generating functions: Recap

Recall that the moment generating function (mgf) of a random variable X is

$$M_X(t) \quad = \quad E[e^{tX}],$$

where t is the variable of the function.

Theorem 1. *The moment generating function M_X of a random variable determines the random variable X.*

2 Product formula for mgf: Theorem

Theorem 2. *Let $X_1, \ldots, X_n$ be independent random variables, let $a_1, \ldots, a_n$ be real numbers, and let*

$$Y \quad = \quad a_1 X_1 + a_2 X_2 + \ldots + a_n X_n.$$

Then the moment generating function of Y is

$$M_Y(t) \quad = \quad M_{X_1}(a_1 t)\, M_{X_2}(a_2 t) \,\ldots\, M_{X_n}(a_n t).$$

Proof: Indeed, this is because

$$\begin{aligned} M_Y(t) \quad &= \quad E[e^{tY}] \quad = \quad E[e^{t(a_1 X_1 + \ldots + a_n X_n)}] \\ &= \quad E[e^{a_1 t X_1} \ldots e^{a_n t X_n}] \quad = \quad E[e^{a_1 t X_1}] \ldots E[e^{a_n t X_n}] \\ &= \quad M_{X_1}(a_1 t) \,\ldots\, M_{X_n}(a_n t). \end{aligned}$$

3 Product formula for mgf: Example

Let X_1 and X_2 be independent random variables, uniform on $\{1, 2\}$, and let $Y = X_1 + 2X_2$.

Compute the probability mass function of Y.

Answer: We can compute the pmf of Y directly, but let's try to do this through mgf:

$$M_Y(t) = M_{X_1}(t)\, M_{X_2}(2t) = \left(\frac{e^t}{2} + \frac{e^{2t}}{2}\right)\left(\frac{e^{2t}}{2} + \frac{e^{4t}}{2}\right) = \frac{e^{3t}}{4} + \frac{e^{4t}}{4} + \frac{e^{5t}}{4} + \frac{e^{6t}}{4}.$$

Hence we conclude that

$$P(Y = 3) = P(Y = 4) = P(Y = 5) = P(Y = 6) = \frac{1}{4}.$$

4 Special case of product formula: Theorem

Theorem 3. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with mgf $M_X(t)$.*
Let Y and $\overline{\mathrm{X}}$ be the random variable

$$Y = X_1 + X_2 + \ldots + X_n, \qquad \overline{\mathrm{X}} = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

Then the mgf of Y and $\overline{\mathrm{X}}$ are given by

$$M_Y(t) = \big[M_X(t)\big]^n, \qquad M_{\overline{\mathrm{X}}}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n.$$

5 Special case of product formula: Exponential to Gamma RVs

Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with mean θ, and let $Y = X_1 + \ldots + X_n$. We already know from Section 3 that Y is a gamma RV with parameters $\alpha = n$ and θ. Let's see if we can derive that using mgf method.

The mgf of exponential RV with mean θ is

$$M_X(t) \quad = \quad \frac{1}{1-\theta t}.$$

Then the mgf for Y is

$$M_Y(t) = [M(t)]^n = \left[\frac{1}{1-\theta t}\right]^n = (1-\theta t)^{-n}.$$

This is the mgf of gamma RV with parameter $\alpha = n$ and θ, so Y is gamma RV with parameter $\alpha = n$ and θ.

The mgf for $\overline{\mathrm{X}}$ is

$$M_{\overline{\mathrm{X}}}(t) = \left[\frac{1}{1-\theta(\frac{t}{n})}\right]^n = \left(1 - \frac{\theta}{n}t\right)^{-n}.$$

This is the mgf of gamma RV with parameter $\alpha = n$ and $\frac{\theta}{n}$, so $\overline{\mathrm{X}}$ is gamma RV with parameter $\alpha = n$ and $\frac{\theta}{n}$.

6 Special case of product formula: Normal RVs

Let $X_1, X_2, \ldots, X_n$ be independent normal RVs with mean $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. The mgf of X is

$$M_{X_i}(t) \quad = \quad e^{\mu_i t + \sigma_i^2 \frac{t^2}{2}} \quad = \quad \exp\big(\mu_i t + \sigma_i^2 \frac{t^2}{2}\big).$$

The mgf of $Y = X_1 + \ldots + X_n$ is

$$\begin{aligned} M_Y(t) \quad &= \quad M_{X_1}(t)\, M_{X_2}(t)\, \ldots\, M_{X_n}(t) \\ = \quad & \exp\big(\mu_1 t + \sigma_1^2 \frac{t^2}{2}\big) \ldots \exp\big(\mu_n t + \sigma_n^2 \frac{t^2}{2}\big) \\ = & \exp\big((\mu_1 + \ldots + \mu_n)t + (\sigma_1^2 + \ldots + \sigma_n^2)\frac{t^2}{2}\big). \end{aligned}$$

Thus Y is the normal RV with mean $\mu_1 + \ldots + \mu_n$ and variances $\sigma_1^2 + \ldots + \sigma_n^2$.

The mgf of $\overline{\mathrm{X}} = \frac{X_1 + \ldots + X_n}{n}$ is

$$\begin{aligned} M_{\overline{\mathrm{X}}}(t) &= M_{X_1}\left(\frac{t}{n}\right) \ldots M_{X_1}\left(\frac{t}{n}\right) \\ &= \exp\left(\mu_1 \frac{t}{n} + \sigma_1^2 \frac{t^2}{2n^2}\right) \ldots \exp\left(\mu_n \frac{t}{n} + \sigma_n^2 \frac{t^2}{2n}\right) \\ &= \exp\left(\frac{\mu_1 + \ldots + \mu_n}{n} t \ + \ \frac{\sigma_1^2 + \ldots + \sigma_n^2}{n^2} \frac{t^2}{2}\right). \end{aligned}$$

Thus $\overline{\mathrm{X}}$ is the normal RV with mean $\frac{\mu_1 + \ldots + \mu_n}{n}$ and variances $\frac{\sigma_1^2 + \ldots + \sigma_n^2}{n^2}$.

Note: In the textbook, this example is instead covered in Section 5.5.

7 Special case of product formula: Chi-square RVs

Let $X_1, \ldots, X_n$ be independent chi-square random variables with r_1, …, r_n degrees of freedom, respectively.

The mgf of $Y = X_1 + \ldots + X_n$ is

$$\begin{aligned} M_Y(t) \quad &= \quad M_{X_1}(t) \ldots M_{X_n}(t) \\ =&(1-2t)^{-r_1/2}\,(1-2t)^{-r_2/2} \ldots (1-2t)^{-r_n/2} \\ =&(1-2t)^{-(r_1+r_2+\ldots+r_n)/2}. \end{aligned}$$

This is the mgf of of chi-square random variable $\chi^2(r_1 + \ldots + r_n)$ with $r_1 + r_2 + \ldots + r_n$ degrees of freedom.

So Y is the random variable $\chi^2(r_1 + \ldots + r_n)$.

8 Normal to chi-square RVs

Let $Z_1, \ldots, Z_n$ be independent standard normal random variables $N(0,1)$, and let W be

$$W = Z_1^2 + Z_2^2 + \ldots + Z_n^2.$$

Recall that Z_i^2 is the chi-square $\chi^2(1)$ random variable with 1 degree of freedom. Hence W is the chi-square $\chi^2(n)$ random variable with n degrees of freedom.

See the textbook for a stronger version that involves normal random variables with general mean and variance.