Math 170E

Lecture Notes Section 5.2 [*,†]

Functions of two random variables

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Image, Two variables: Definition

Let S_X be a two dimensional region.

Let $u_1(\cdot,\cdot) : S_X \to \mathbb{R}$ and $u_2(\cdot,\cdot) : S_X \to \mathbb{R}$ be two functions.

The **image** S_Y is

$$S_Y = \left\{ \big(u_1(x_1,x_2), u_2(x_1,x_2)\big) \mid (x_1,x_2) \in S_X \right\}.$$

Note: Usually we find S_Y by drawing pictures.

Note: In this notes, elements of S_X are denoted by (x_1, x_2), and elements of S_Y are denoted by (y_1, y_2).

2 Image, Two variables: Example

Let S_X be the region

$$S_X = \{(x_1, x_2) \mid 0 < x_1 < x_2 < 1\}.$$

Let $u_1(\cdot, \cdot)$ and $u_2(\cdot, \cdot)$ be functions

$$u_1(x_1, x_2) = \frac{x_1}{x_2}; \qquad u_2(x_1, x_2) = x_2,$$

for all $(x_1, x_2) \in S_X$.

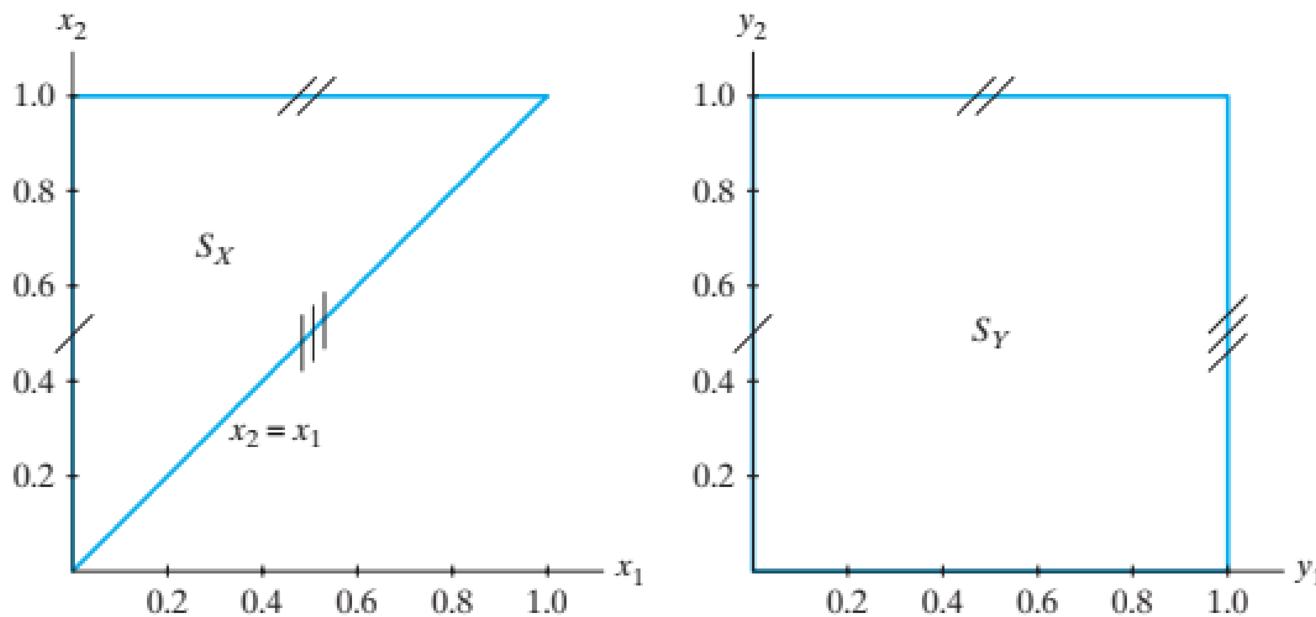
Find the image S_Y.

3 Image, Two variables: Answer

The image S_Y is equal to

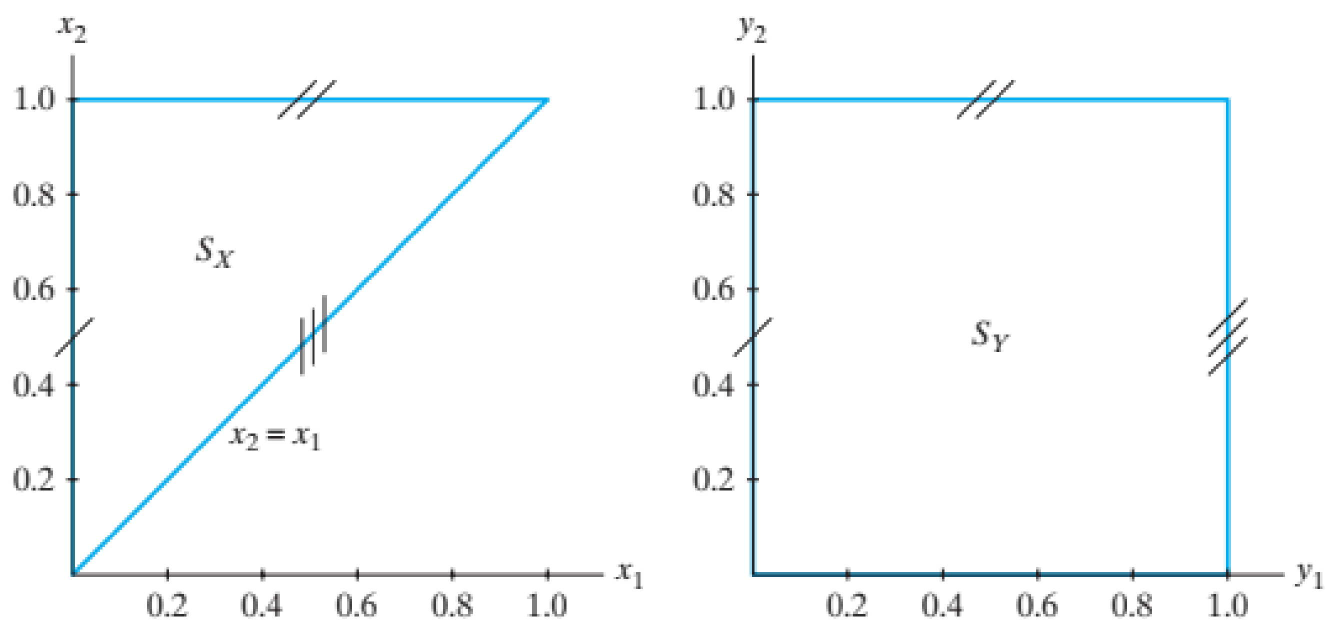
$$S_Y = \{(y_1, y_2) \mid 0 < y_1, y_2 < 1\}.$$

(See the picture.)



4 Invertible functions: Two variables

The functions u_1, u_2 that maps S_X to S_Y are **invertible** if, for any point (y_1, y_2) in S_Y, there is a unique (x_1, x_2) in S_X such that u_1, u_2 maps (x_1, x_2) to (y_1, y_2).



Note: For simplicity, all functions u_1 and u_2 in this lecture note are invertible.

5 Inverse functions: Two variables

Let u_1 and u_2 be invertible functions.

The inverse functions $v_1(\cdot,\cdot) : S_Y \to \mathbb{R}$ and $v_2(\cdot,\cdot) : S_Y \to \mathbb{R}$ are functions such that

- For all $(x_1, x_2) \in S_X$:

$$v_1(u_1(x_1, x_2)) = x_1; \qquad v_2(u_2(x_1, x_2)) = x_2.$$

- For all $(y_1, y_2) \in S_Y$:

$$u_1(v_1(y_1, y_2)) = y_1; \qquad u_2(v_2(y_1, y_2)) = y_2.$$

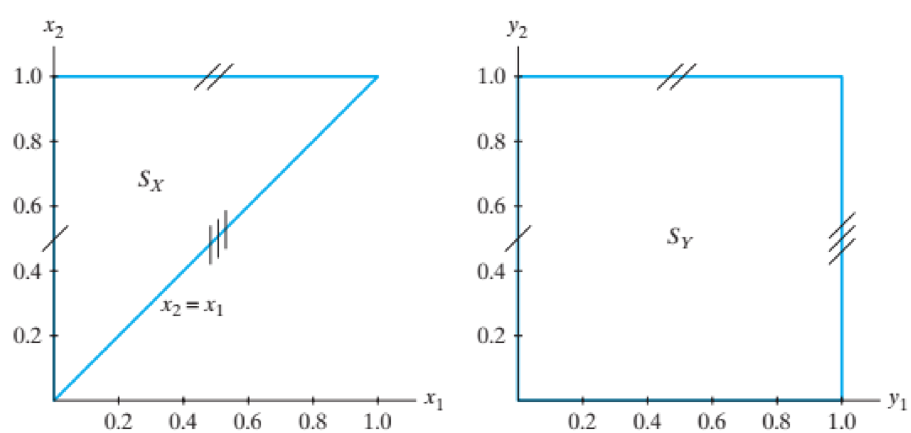
6 Inverse functions: Example

Let S_X be u_1 and u_2 be from the previous example:

$$S_X = \{(x_1, x_2) \mid 0 < x_1 < x_2 < 1\},$$
$$u_1(x_1, x_2) = \frac{x_1}{x_2}; \qquad u_2(x_1, x_2) = x_2.$$

The image S_Y and the inverse functions are given by

$$S_Y = \{(y_1, y_2) \mid 0 < y_1, y_2 < 1\},$$
$$v_1(y_1, y_2) = y_1 y_2; \qquad v_2(y_1, y_2) = y_2.$$



7 How to find inverse functions

1. The input is

$$\begin{aligned} u_1 &= \text{an expression in terms of } x_1 \text{ and } x_2; \\ u_2 &= \text{another expression in terms of } x_1 \text{ and } x_2. \end{aligned}$$

2. Make the following substitution: u_1 becomes y_1, u_2 becomes y_2, x_1 becomes v_1, x_2 becomes v_2.

$$\begin{aligned} y_1 &= \text{an expression in terms of } v_1 \text{ and } v_2; \\ y_2 &= \text{another expression in terms of } v_1 \text{ and } v_2. \end{aligned}$$

3. Solve the equations, and write v_1, v_2 as expressions in terms of y_1 and y_2.

8 How to find inverse functions: Example

Let $u_1(\cdot,\cdot)$ and $u_2(\cdot,\cdot)$ be the functions from the previous example:

$$u_1(x_1, x_2) = \frac{x_1}{x_2}; \qquad u_2(x_1, x_2) = x_2,$$

for $x_1, x_2 \in S_X$.

Compute the inverse functions $v_1(\cdot,\cdot)$ and $v_2(\cdot,\cdot)$.

9 How to find inverse functions: Answer

We are given

$$u_1 = \frac{x_1}{x_2}; \qquad u_2 = x_2.$$

Make the following substitution: u_1 becomes y_1, u_2 becomes y_2, x_1 becomes v_1, x_2 becomes v_2,

$$y_1 \;=\; \frac{v_1}{v_2}; \qquad y_2 = v_2.$$

Solving for v_1 and v_2, we get

$$v_1 \;=\; y_1 y_2; \qquad v_2 \;=\; y_2.$$

Thus the inverse function is

$$v_1(y_1, y_2) \;=\; y_1 y_2; \qquad v_2(y_1, y_2) \;=\; y_2.$$

10 Functions of two random variables

We want to solve the following problem:

- Input: Two random variables X_1, X_2 and **invertible** functions u_1, u_2.

- Output: New random variables $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$. (i.e., derive the pmf, pdf, or cdf of Y).

11 COV, two RVs: Theorem

Let X_1 and X_2 be two RVs with joint support S_X and joint pdf $f(\cdot,\cdot)$.

Let $u_1(\cdot,\cdot)$ and $u_2(\cdot,\cdot)$ be two invertible functions.

Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$.

The joint support and pdf of Y_1,Y_2 can be computed by

1. Compute the inverse $v_1(\cdot,\cdot)$ and $v_2(\cdot,\cdot)$,
2. The joint support of Y is the image S_Y.
3. Compute the Jacobian

$$J = \det \begin{bmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{bmatrix}.$$

4. The joint pdf $g(\cdot,\cdot)$ of Y_1 and Y_2 are then given by

$$g(y_1, y_2) \quad = \quad \big|J\big| \; f\big(v_1(y_1, y_2), v_2(y_1, y_2)\big).$$

12 COV, two RVs: Example

X_1 and X_2 are RVs with support and joint pdf

$$\begin{aligned} S_X =& \{(x_1, x_2) \mid 0 < x_1 < x_2 < 1.\} \\ f(x_1, x_2) =& 2 \quad \text{for } (x_1, x_2) \in S_X. \end{aligned}$$

The functions u_1 and u_2 are given by

$$u_1(x_1, x_2) = \frac{x_1}{x_2}; \qquad u_2(x_1, x_2) = x_2.$$

The outputs are

$$Y_1 = u_1(X_1, X_2) = \frac{X_1}{X_2}; \qquad Y_2 = u_2(X_1, X_2) = X_2.$$

Compute the joint support and the joint pdf of Y_1 and Y_2.

13 COV, two RVs: Answer

We already compute the inverse functions from before:

$$v_1(y_1, y_2) = y_1 y_2; \qquad v_2(y_1, y_2) = y_2.$$

The support of Y_1 and Y_2 is the image S_Y:

$$S_Y = \{(y_1, y_2) \mid 0 < y_1, y_2 < 1\}.$$

The Jacobian is

$$J = \det \begin{bmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2.$$

This joint pdf is then given by

$$\begin{aligned} g(y_1, y_2) &= |J| \; f\big(v_1(y_1, y_2), v_2(y_1, y_2)\big) \\ &= 2y_2 \qquad 0 < y_1, y_2 < 1. \end{aligned}$$

14 COV, two RVs: Laplace Example

Let X_1 and X_2 be independent exponential random variables with mean 1,

$$\begin{aligned} S_X &= (0,\infty) \times (0,\infty) \\ f(x_1, x_2) &= e^{-x_1 - x_2}, \qquad 0 < x_1, x_2 < \infty. \end{aligned}$$

Compute the joint support, joint pdf, and the marginal pdfs of

$$Y_1 = X_1 - X_2; \qquad Y_2 = X_1 + X_2.$$

15 COV, two RVs: Laplace Answer

The function u_1 and u_2 are given by

$$u_1(x_1, x_2) = x_1 - x_2; \qquad u_2(x_1, x_2) = x_1 + x_2.$$

Make the following substitution: u_1 becomes y_1, u_2 becomes y_2, x_1 becomes v_1, x_2 becomes v_2,

$$y_1 = v_1 - v_2 \tag{1}$$

$$y_2 = v_1 + v_2. \tag{2}$$

Adding (1) to (2), then divide by 2, we get

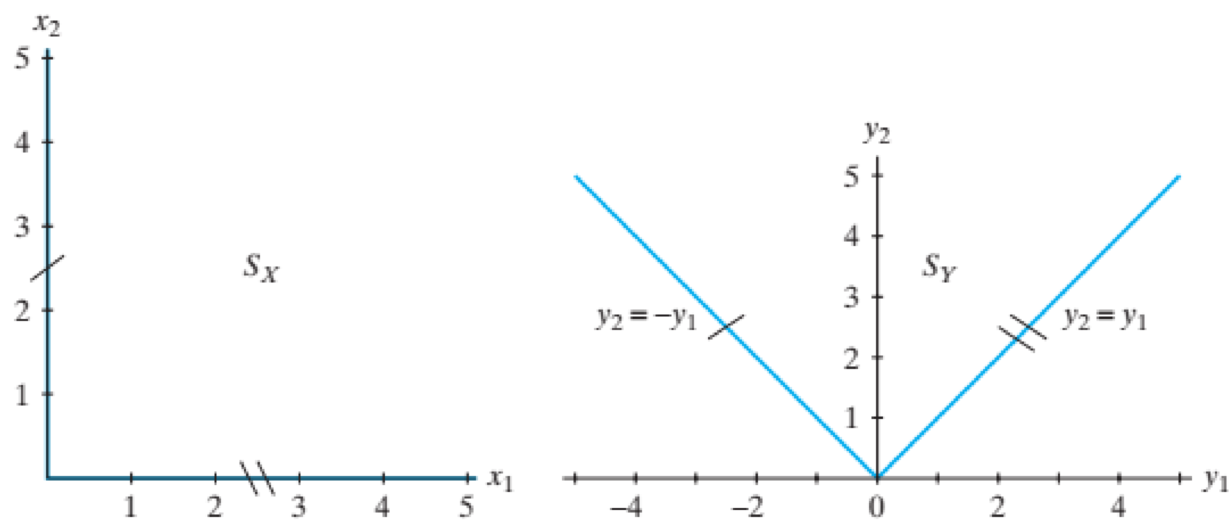
$$v_1(y_1, y_2) = \frac{y_1 + y_2}{2}.$$

Subtract (1) from (2), then divide by 2, we get

$$v_2(y_1, y_2) \quad = \quad \frac{y_2 - y_1}{2}.$$

The support is given by (see picture)

$$S_Y = \{(y_1, y_2) \mid -y_2 < y_1 < y_2,\ 0 < y_2 < \infty\}.$$



We now compute the Jacobian:

$$J = \det \begin{bmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}.$$

The joint pdf is then given by

$$\begin{aligned} g(y_1, y_2) &= |J| f\big(v_1(y_1, y_2), v_2(y_1, y_2)\big) \\ &= \left|\frac{1}{2}\right| \left(\exp\left[-\left(\frac{y_1 + y_2}{2}\right) - \left(\frac{y_2 - y_1}{2}\right)\right]\right) \\ &= \frac{1}{2} e^{-y_2} \qquad - y_2 < y_1 < y_2,\ 0 < y_2 < \infty. \end{aligned}$$

The marginal pdf for Y_1 is then given by

$$
\begin{aligned}
g_1(y_1) &= \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_2 &= \int_{|y_1|}^{\infty} \frac{e^{-y_2}}{2}\, dy_2 \\
&= \left[-\frac{e^{-y_2}}{2} \right]_{|y_1|}^{\infty} &= \frac{1}{2} e^{-|y_1|} \qquad -\infty < y_1 < \infty;
\end{aligned}
$$

This function is called **double exponential**, or **Laplace pdf**.

The marginal pdf for Y_2 is then given by

$$
\begin{aligned}
g_2(y_2) &= \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_1 = \int_{-y_2}^{y_2} \frac{e^{-y_2}}{2}\, dy_1 = \left[\frac{y_1 e^{-y_2}}{2} \right]_{y_1=-y_2}^{y_1=y_2} \\
&= y_2 e^{-y_2} \qquad 0 < y_2 < \infty.
\end{aligned}
$$

There is another example in textbook where X_1 and X_2 are independent gamma distribution and Y_1 and Y_2 are given by

$$Y_1 = \frac{X_1}{X_1 + X_2}; \qquad Y_2 = X_1 + X_2.$$

Try to solve it using the method above as an exercise.

16 COV, two RVs: Box-Muller Example

Let X_1 and X_2 be independent uniform random variables on the interval $[0, 1]$. Let

$$Y_1 = \sqrt{-2 \ln X_1} \cos(2\pi X_2); \qquad Y_2 = \sqrt{-2 \ln X_1} \sin(2\pi X_2).$$

Compute the joint pdf of Y_1 and Y_2, and the marginal pdf of Y_1 and Y_2.

17 COV, two RVs: Box-Muller Answer

X_1 and X_2 have joint pdf and support

$$\begin{aligned} S_X &= \{(x_1, x_2) \mid 0 < x_1, x_2 < 1\}; \\ f(x_1, x_2) &= 1, \qquad \text{for } (x_1, x_2) \in S_X. \end{aligned}$$

The functions u_1 and u_2 are given by

$$\begin{aligned} u_1(x_1, x_2) &= \sqrt{-2 \ln x_1} \cos(2\pi x_2); \\ u_2(x_1, x_2) &= \sqrt{-2 \ln x_1} \sin(2\pi x_2). \end{aligned}$$

Make the following substitution: u_1 becomes y_1, u_2 becomes y_2, x_1 becomes v_1, x_2 becomes v_2,

$$y_1 = \sqrt{-2\ln v_1}\cos(2\pi v_2); \tag{3}$$

$$y_2 = \sqrt{-2\ln v_1}\sin(2\pi v_2). \tag{4}$$

Squaring (3) and (4), then adding them together, we get

$$\begin{aligned}
(y_1)^2 + (y_2)^2 =& (-2\ln v_1)(\cos^2(\pi v_2) + \sin^2(\pi v_2)) \\
(y_1)^2 + (y_2)^2 =& -2\ln v_1 \\
v_1 =& \exp\left(-\frac{(y_1)^2 + (y_2)^2}{2}\right).
\end{aligned}$$

Divide (4) by (3), we get

$$
\begin{aligned}
\frac{y_2}{y_1} &= \frac{\sqrt{-2\ln v_1}\sin(2\pi v_2)}{\sqrt{-2\ln v_1}\cos(2\pi v_2)} \\
\frac{y_2}{y_1} &= \tan(2\pi v_2) \\
v_2 &= \frac{1}{2\pi}\arctan\left(\frac{y_2}{y_1}\right).
\end{aligned}
$$

The support is computed by picture in the lecture using polar coordinates (watch the recording),

$$
S_Y \quad = \quad \{(y_1, y_2) \mid -\infty < y_1, y_2 < \infty\}.
$$

The Jacobian is given by

$$
\begin{aligned}
J =& \det \begin{bmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{bmatrix} \\
=& \det \begin{bmatrix} -y_1 \exp\left(-\frac{(y_1)^2+(y_2)^2}{2}\right) & -y_2 \exp\left(-\frac{(y_1)^2+(y_2)^2}{2}\right) \\ \frac{-y_2}{2\pi[(y_1)^2+(y_2)^2]} & \frac{y_1}{2\pi[(y_1)^2+(y_2)^2]} \end{bmatrix} \\
=& \exp\left(-\frac{(y_1)^2+(y_2)^2}{2}\right) \frac{1}{2\pi[(y_1)^2+(y_2)^2]} \; \det \begin{bmatrix} -y_1 & -y_2 \\ \\ -y_2 & y_1 \end{bmatrix} \\
=& \exp\left(-\frac{(y_1)^2+(y_2)^2}{2}\right) \frac{-(y_1)^2-(y_2)^2}{2\pi[(y_1)^2+(y_2)^2]} \\
=& \frac{-1}{2\pi} \exp\left(-\frac{(y_1)^2+(y_2)^2}{2}\right).
\end{aligned}
$$

The joint pdf is then given by

$$
\begin{aligned}
g(y_1, y_2) =& |J| f\left(v_1(y_1, y_2), v_2(y_1, y_2)\right) \\
=& \left| \frac{-1}{2\pi} \exp\left(-\frac{(y_1)^2 + (y_2)^2}{2}\right) \right| \; (1) \\
=& \frac{1}{2\pi} \exp\left(-\frac{(y_1)^2 + (y_2)^2}{2}\right).
\end{aligned}
$$

We can compute the marginal pdf of Y_1 and Y_2 separately. However, a faster way is to see that the joint pdf of Y_1 and Y_2 is the pdf for standard normal bivariate distribution. This implies that both Y_1 and Y_2 are standard normal random variables, so the pdfs are

$$
g_1(y) = g_2(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y)^2}{2}\right).
$$

18 Cdf method: Example

Here we demonstrate another method to compute two-variable random variables transformation by using cumulative distributive functions.

Let X_1 and X_2 be independent chi-square random variables with 2 degrees of freedom.
Let Y be the random variable

$$Y = \frac{X_1}{X_2}.$$

Compute the pdf of Y.

19 Cdf method: answer

Answer: Note that the joint pdf and the joint support of X_1 and X_2 are given by

$$\begin{aligned} S_X &= \{(x_1, x_2) \mid 0 < x_1, x_2 < \infty\}; \\ f(x_1, x_2) &= \left(\frac{1}{2}e^{-x_1/2}\right)\left(\frac{1}{2}e^{-x_2/2}\right) = \frac{1}{4}\exp\left(-\frac{x_1+x_2}{2}\right). \end{aligned}$$

We now compute the cdf of Y.

$$\begin{aligned} F(y) =& P(Y \le y) = P\left(\frac{X_1}{X_2} \le y\right) = P(X_1 \le yX_2) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{yx_2} f(x_1, x_2)\, dx_1\, dx_2 \\ &= \int_0^{\infty}\int_0^{yx_2} \frac{1}{4}\exp\left(-\frac{x_1+x_2}{2}\right)\, dx_1\, dx_2. \end{aligned}$$

Trying to integrate the equation above is possible, but it will be very tedious! Instead, let's move on and compute the pdf.

$$
\begin{aligned}
f(y) &= \frac{\partial}{\partial y} F(y) = \frac{\partial}{\partial y}\left(\int_0^{\infty}\int_0^{yx_2} \frac{1}{4}\exp\left(-\frac{x_1+x_2}{2}\right)\,dx_1\,dx_2\right) \\
&= \int_0^{\infty} \frac{\partial}{\partial y}\left(\int_0^{yx_2} \frac{1}{4}\exp\left(-\frac{x_1+x_2}{2}\right)\,dx_1\right)\,dx_2 \\
&= \int_0^{\infty} (x_2)\,\frac{1}{4}\exp\left(-\frac{(yx_2)+x_2}{2}\right)\,dx_2 \\
&= \frac{1}{4}\int_0^{\infty} x_2\,\exp\left(-\frac{1+y}{2}x_2\right)\,dx_2 \\
&= \frac{1}{4}\left[x_2\frac{-2}{1+y}\exp\left(-\frac{1+y}{2}x_2\right)\right]_{x_2=0}^{x_2=\infty} \\
&\quad - \frac{1}{4}\int_0^{\infty} \frac{-2}{1+y}\exp\left(-\frac{1+y}{2}x_2\right)\,dx_2 \\
&= 0 \,-\, \frac{1}{4}\left[\frac{4}{(1+y)^2}\exp\left(-\frac{1+y}{2}x_2\right)\right]_{x_2=0}^{x_2=\infty} \\
&= \frac{1}{(1+y)^2} \qquad y > 0.
\end{aligned}
$$

As an exercise, read textbook for a proof of the general case when X_1 and X_2 have arbitrary degrees of freedom.