Math 170E

Lecture Notes Section 5.1 *†

Functions of a random variable

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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†This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Functions of a random variable

We want to solve the following problem:

- Input: A random variable X and a function u.
- Output: A new random variable $Y = u(X)$. (i.e., derive the pmf, pdf, or cdf of Y).

2 Functions of an RV: Example

Let X be a discrete uniform distribution on $\{-2, -1, 0, 1, 2\}$.

Let $Y = X^2$. Find the probability mass function of Y.

Answer: The pmf f of x is given by

$$f(x) = P(X = x) = \frac{1}{5} \quad \text{for } x \in \{-2, -1, 1, 0, 2\}.$$

Since $Y = X^2$, the support of Y is $\{0, 1, 4\}$, and the pmf $g(y)$ of Y is

$$\begin{aligned}
g(0) =& P(Y = 0) = P(X^2 = 0) = P(X = 0) = \frac{1}{5};\\
g(1) =& P(Y = 1) = P(X^2 = 1) = P(X = -1) + P(X = 1) = \frac{2}{5};\\
g(4) =& P(Y = 4) = P(X^2 = 4) = P(X = -2) + P(X = 2) = \frac{2}{5}.
\end{aligned}$$

3 Inverse function: Definition

Let $u : S_X \to S_Y$ be a function.

The *inverse function* $v : S_Y \to S_X$ is the function such that

$$
\begin{aligned}
v(u(x)) = & x \quad \text{for all } x \in S_X; \text{ and} \\
u(v(y)) = & y \quad \text{for all } y \in S_Y.
\end{aligned}
$$

4 Inverse function: Example 1

Let that S_X and S_Y be the set of integers,

$$S_X = S_Y = \{\ldots, -1, 0, 1, \ldots\}.$$

Let $u : S_X \to S_Y$ be the function

$$u(x) = 1 + x, \quad x \in \{\ldots, -1, 0, 1, \ldots\}.$$

Then the inverse function $v : S_Y \to S_X$ is given by

$$v(y) = y - 1, \quad y \in \{\ldots, -1, 0, 1, \ldots\}.$$

To see this, note that

$$\begin{aligned} v(u(x)) =& v(1 + x) = (1 + x) - 1 = x; \\ u(v(y)) =& u(y - 1) = 1 + (y - 1) = y. \end{aligned}$$

5 Inverse function: Example 2

Let S_X and S_Y be the set of positive real numbers,

$$S_X = S_Y = \{x \in \mathbb{R} \mid x \geq 0\}.$$

Let $u : S_X \to S_Y$ be the function

$$u(x) = x^2, \qquad x \geq 0.$$

Then the inverse function $v : S_Y \to S_X$ is the function

$$v(y) = \sqrt{y}, \qquad y \geq 0.$$

To see this, note that

$$\begin{aligned} v(u(x)) =& v(x^2) = \sqrt{(x^2)} = |x| = x; \\ u(v(y)) =& u(\sqrt{y}) = (\sqrt{y})^2 = |y| = y. \end{aligned}$$

6 Inverse function: Non-example

Let S_X and S_Y be given by

$$S_X := \mathbb{R}; \quad S_Y = \{x \in \mathbb{R} \mid x \geq 0\}.$$

Let $u : S_X \to S_Y$ be the function from before

$$u(x) = x^2, \qquad x \geq 0.$$

It might be tempting to say that the $v : S_Y \to S_X$ is the inverse function for u:

$$v(y) = \sqrt{y}, \qquad y \geq 0.$$

However, this is not the case, since, for all negative x,

$$v(u(x)) = v(x^2) = \sqrt{(x^2)} = |x| = -x.$$

7 COV, discrete invertible: Theorem

The input is

- A **discrete** RV X with support $S_X = \{c_1, c_2, \ldots\}$ and pmf f; and
- An **invertible** function $u : S_X \to S_Y$.

Then the output $Y = u(X)$ can be computed by the three-steps method:

1. The support S_Y is $\{u(c_1), u(c_2), \ldots\}$.
2. Compute the inverse function $v : S_Y \to S(x)$.
3. The pmf $g(y)$ of Y is given by

$$g(y) = f(v(y)) \quad \text{for all } y \in S_Y.$$

8 Change-of-variable, discrete invertible: Example

Let X be the Poisson random variable with $\lambda = 4$,

$$S_X = \{0, 1, 2, \ldots\}; \qquad f(x) = \frac{4^x e^{-4}}{x!}, \qquad x \in \{0, 1, 2, \ldots\}.$$

Let $u : S_X \to S_Y$ be the function

$$u(x) = \sqrt{x}, \qquad x \in \{0, 1, 2, \ldots\}.$$

Compute the support and pmf of $Y = u(X)$.

9 COV, discrete invertible: Answer

The support S_Y of $Y = \sqrt{X}$ is equal to

$$S_Y = \{0, 1, \sqrt{2}, \sqrt{3}, \ldots\}.$$

The inverse function $v : S_Y \to S_X$ is

$$v(y) = y^2, \qquad y \in \{0, 1, \sqrt{2}, \sqrt{3}, \ldots\}.$$

So the pmf of Y is

$$\begin{aligned} g(y) &= P(Y = y) = P(u(X) = y) \\ &= P(X = v(y)) = f(v(y)) \\ &= f(y^2) = \frac{4^{y^2} e^{-4}}{(y^2)!}, \qquad y \in \{0, 1, \sqrt{2}, \sqrt{3}, \ldots\}. \end{aligned}$$

10 COV, continuous invertible: Theorem

Let X be a **continuous** RV with $S_X = \{x \mid c_1 < x < c_2\}$ and pdf f.

Let $u : S_X \to S_Y$ be a continuous **invertible** function.

Then $Y = u(X)$ can be computed by:

1. The support S_Y is the interval of $\mathbb{R}$ bounded by $u(c_1)$ and $u(c_2)$.

2. Compute the inverse function $v : S_Y \to S_X$ of u.

3. Compute the derivative v' of v.

4. The pdf $g(y)$ of Y is given by

$$g(y) = f\big(v(y)\big)\,\big|v'(y)\big|.$$

11 COV, continuous invertible: Example 1

Let X be the exponential random variable with $\theta = 1$, so

$$\begin{aligned} S_X =& \mathbb{R}_{>0} = \{x \in \mathbb{R} \mid 0 < x < \infty\}; \\ f(x) =& e^{-x}; \qquad 0 < x < \infty. \end{aligned}$$

Let $u : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be the function

$$u(x) = e^x \qquad 0 < x < \infty.$$

Compute the support and the pdf of $Y = u(X)$.

12 COV, continuous invertible: Answer 1

The support of Y is

$$\begin{aligned} S_Y &= \{u(x) \mid x \in S_X\} &= \{e^x \mid 0 < x < \infty\} \\ &= \{y \in \mathbb{R} \mid 1 < y < \infty\}. \end{aligned}$$

The inverse function $v : S_Y \to S_X$ is

$$v(y) = \ln y, \qquad y \in (1, \infty).$$

The cdf of Y is then equal to

$$\begin{aligned} G(y) =& P(Y \leq y) = P(u(X) \leq y) = P(X \leq v(y)) \\ =& P(X \leq \ln y) = \int_0^{\ln y} e^{-x}\, dx. \end{aligned}$$

Since the pdf is the derivative of the cdf, we have

$$
\begin{aligned}
g(y) = G'(y) &= \frac{\partial}{dy}\left(\int_0^{\ln y} e^{-x}\,dx\right) \\
&= (\ln y)' e^{-\ln y} \qquad \text{(Fundamental Theorem of Calculus)} \\
&= \left(\frac{1}{y}\right)\left(\frac{1}{y}\right) = \frac{1}{y^2}.
\end{aligned}
$$

13 COV, continuous invertible: Example 2

Let X be be the uniform random variable on $(-\pi/2, \pi/2)$:

$$
\begin{aligned}
S_X =& (-\pi/2, \pi/2) = \{x \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\};\\
f(x) =& \frac{1}{\pi} \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.
\end{aligned}
$$

Let $u : (-\pi/2, \pi/2) \to \mathbb{R}$ be the (continuous invertible) function

$$u(x) = -\tan x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Compute the support and the pdf of $Y = u(X)$.

14 COV, continuous invertible: Example 2

We have

$$u(c_1) = -\tan(-\pi/2) = \infty; \qquad u(c_2) = -\tan(\pi/2) = -\infty,$$

so the support of Y is $S_Y = (-\infty, \infty) = \mathbb{R}$.

The inverse function $v : \mathbb{R} \to (-\pi/2, \pi/2)$ is given by

$$v(y) \quad = \quad -\arctan y.$$

The derivative v' is then given by

$$v'(y) \quad = \quad \frac{-1}{1+y^2}.$$

The pdf of Y is then given by

$$g(y) \quad = \quad f\big(v(y)\big)\left|v'(y)\right| \quad = \quad \frac{1}{\pi}\left|\frac{-1}{1+y^2}\right| \quad = \quad \frac{1}{\pi(1+y^2)}.$$

As exercises, read the lognormal RV and Cauchy RV example in the textbook.

15 COV, two-to-one: Example

Let X be the random variable

$$S_X = \{x \mid -1 < x < 2\};$$
$$f(x) = \frac{x^2}{3}, \qquad -1 < x < 2.$$

Let $u : S_X \to S_Y$ is the function

$$u(x) = x^2, \qquad -1 < x < 2.$$

Compute the support and the pdf of $Y = u(X)$.

16 COV, two-to-one: Answer

Since u is not invertible, we **cannot** compute the inverse function directly.

To fix this issue, we split S_X into $S_{X_1} \cup S_{X_2}$, where

$$S_{X_1} = \{x \mid -1 < x < 0\}; \qquad S_{X_2} = \{x \mid 0 < x < 2\}.$$

Applying u to these two supports, we have

$$\begin{aligned} S_{Y_1} &= u(S_{X_1}) &= \{y \mid 0 < y < 1\}; \\ S_{Y_2} &= u(S_{X_2}) &= \{y \mid 0 < y < 4\}. \end{aligned}$$

So the support of Y is equal to

$$S_Y = S_{Y_1} \cup S_{Y_2} = \{y \mid 0 < y < 4\}.$$

Note that u is invertible on S_{X_1}, with the inverse $v_1 : S_{Y_1} \to S_{X_1}$ given by

$$v_1(y) = -\sqrt{y}, \qquad 0 < y < 1.$$

Also note that u is invertible on S_{X_2}, with the inverse $v_2 : S_{Y_2} \to S_{X_2}$ given by

$$v_2(y) = \sqrt{y}, \qquad 0 < y < 4.$$

The derivatives of v_1 and v_2 are

$$v_1'(y) = \frac{-1}{2\sqrt{y}}, \qquad v_2'(y) = \frac{1}{2\sqrt{y}}.$$

We can now apply COV to each case separately,

$$\begin{aligned}
g_1(y) &= f\big(v_1(y)\big)\left|v_1'(y)\right| \quad = \quad \frac{(-\sqrt{y})^2}{3}\left|\frac{-1}{2\sqrt{y}}\right| \\
&= \frac{\sqrt{y}}{6} \quad \text{with support } 0 < y < 1\,.
\end{aligned}$$

$$\begin{aligned}
g_2(y) &= f\big(v_2(y)\big)\left|v_2'(y)\right| \quad = \quad \frac{(\sqrt{y})^2}{3}\left|\frac{1}{2\sqrt{y}}\right| \\
&= \frac{\sqrt{y}}{6} \quad \text{with support } 0 < y < 4\,.
\end{aligned}$$

It then follows that the pdf g of Y is given by

$$g(y) = g_1(y) + g_2(y) = \begin{cases} \frac{\sqrt{y}}{6} + \frac{\sqrt{y}}{6} = \frac{\sqrt{y}}{3}, & 0 < y < 1; \\ \frac{\sqrt{y}}{6} & 1 < y < 4. \end{cases}$$

17 COV, two-to-one: Theorem

Let X be a **continuous** random variable, and let $u : S_X \to S_Y$ be a **non-invertible** continuous function. Then $Y = u(X)$ can (possibly) be computed by

1. Split the support S_X into $S_{X_1} \cup S_{X_2}$ so that $u_1 : S_{X_1} \to S_{Y_1}$ and $u_2 : S_{X_2} \to S_{Y_2}$ are invertible.

2. Compute the inverse function $v_1 : S_{Y_1} \to S_{X_1}$ and $v_2 : S_{Y_2} \to S_{X_2}$.

3. Compute $g_1(y)$ and $g_2(y)$ by

$$\begin{aligned} g_1(y) &= f\big(v_1(y)\big)\,\big|v_1'(y)\big| \quad \text{with support} \quad y \in S_{Y_1}; \\ g_2(y) &= f\big(v_2(y)\big)\,\big|v_2'(y)\big| \quad \text{with support} \quad y \in S_{Y_2}. \end{aligned}$$

4. The random variable Y is given by

$$
\begin{aligned}
S_Y &= S_{Y_1} \cup S_{Y_2}; \\
g(y) &= g_1(y) + g_2(y) \qquad y \in S_y.
\end{aligned}
$$

Remark: Sometimes the function u cannot be split into two invertible functions. In that case, modify Step 1 so that u is instead split into k invertible functions, for large enough $k \geq 3$.